MATHEMATICAL FRONTIERS

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Mathematical Sciences & Analytics

MATHEMATICAL FRONTIERS
2019 Monthly Webinar Series, 2-3pm ET

February 12: *Machine Learning for Materials Science**

March 12: *Mathematics of Privacy**

April 9: *Mathematics of Gravitational Waves**

May 14: *Algebraic Geometry*

June 11: *Mathematics of Transportation*

July 9: *Cryptography & Cybersecurity*

August 13: *Machine Learning in Medicine*

September 10: *Logic and Foundations*

October 8: *Mathematics of Quantum Physics*

November 12: *Quantum Encryption*

December 10: *Machine Learning for Text*

Made possible by support for BMSA from the
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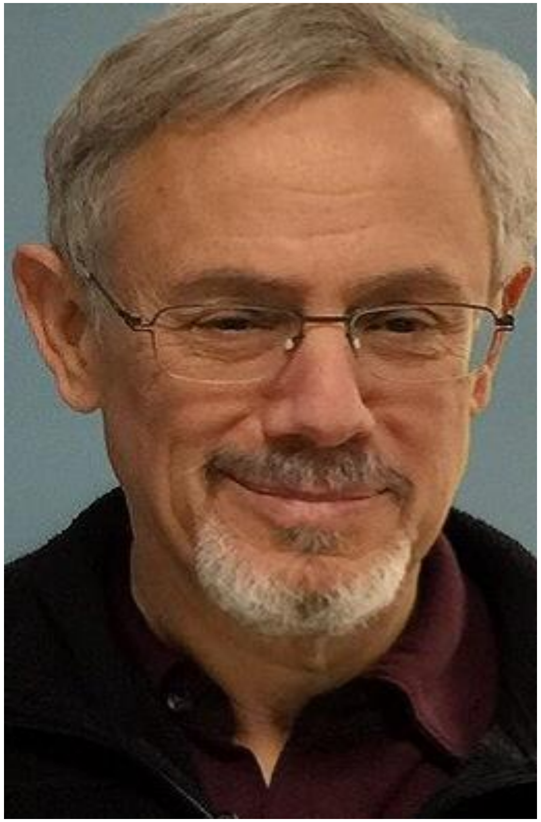
MATHEMATICAL FRONTIERS
Algebraic Geometry





Ravi Vakil,
Stanford University

Bernd Sturmfels,
UC Berkeley

Mark Green,
UCLA (moderator)

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MATHEMATICAL FRONTIERS
Algebraic Geometry

Ravi Vakil,
Stanford University

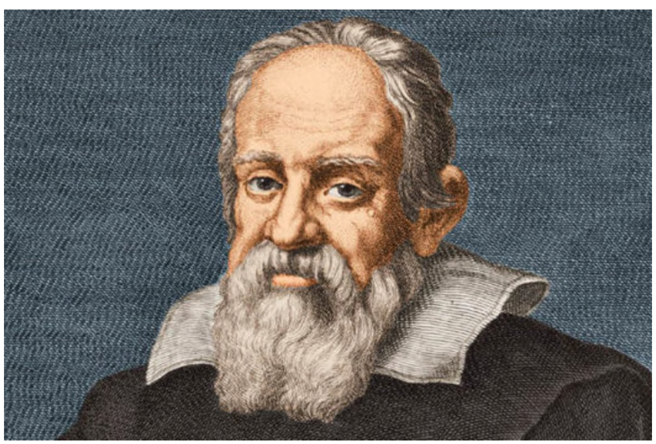
Professor of Mathematics,
Robert K. Packard University Fellow

What is Algebraic Geometry?

What is algebraic geometry?

To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty of nature ... If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in.

- Feynman, the character of physical law (1965)



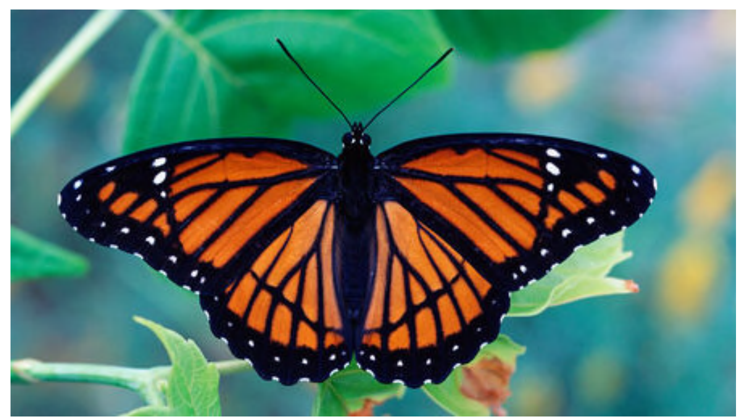
(Natural) Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth.

-Galileo Galilei, in The Assayer, 1623

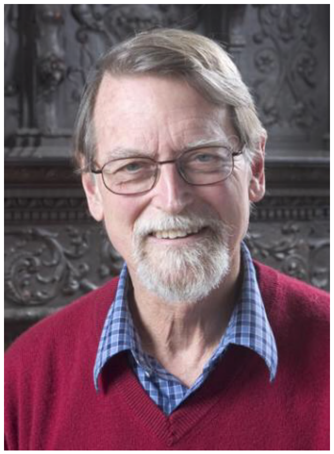
What is algebraic geometry?

before linear algebra

after learning linear algebra



What is algebraic geometry?



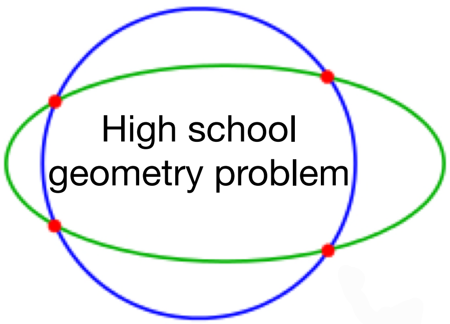
Algebraic geometry “seems to have acquired the reputation of being esoteric, exclusive, and very abstract, with adherents who are secretly plotting to take over all the rest of mathematics! In one respect this last point is accurate …”

- David Mumford

before learning algebraic geometry

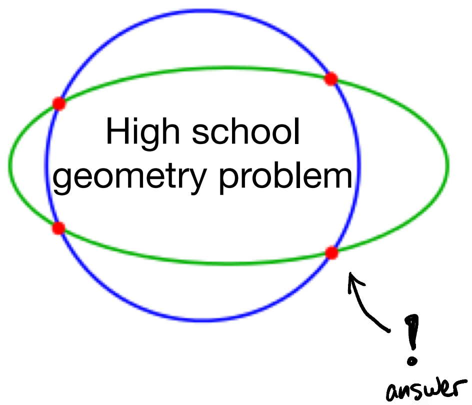
after learning algebraic geometry

What is algebraic geometry?



geometry

What is algebraic geometry?



$$(x-3)^2 + (y-2)^2 = 40$$
$$3(x-9)^2 + 2(x+1)^2 = 3$$
$$2y + 9x = 4$$

solve

$$x = 2, \quad y = 3$$

geometry

Cartesian coordinates

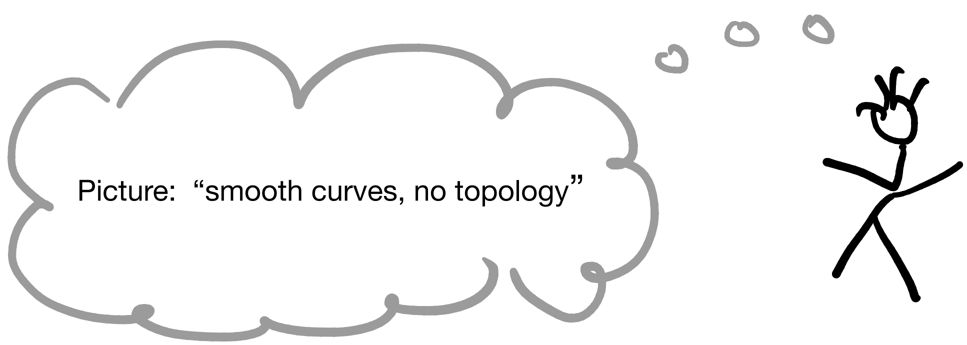
algebra

What is algebraic geometry?

polynomials	$\sim$	integers
e.g. $3x^2 + 2$	strangely similar	e.g. 34

both have "unique factorization"

$x^5 + 2x^3 + x = (x^2+1) \cdot x$ $\qquad$ $12 = 2^2 \times 3$



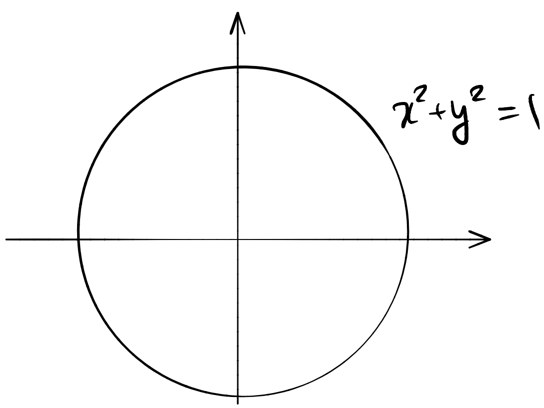
What is algebraic geometry?

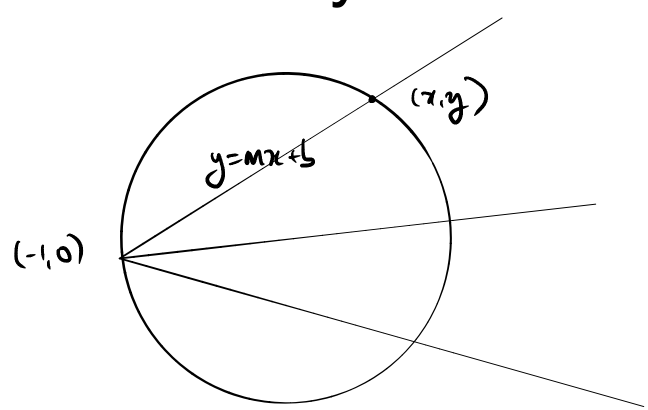
Find integer solutions to $a^2 + b^2 = c^2$

Pythagoras

"Plimpton 322"
(Babylonian clay tablet,
~ 1800 BCE)

Find *rational* solutions to $x^2 + y^2 = 1$





What is algebraic geometry?

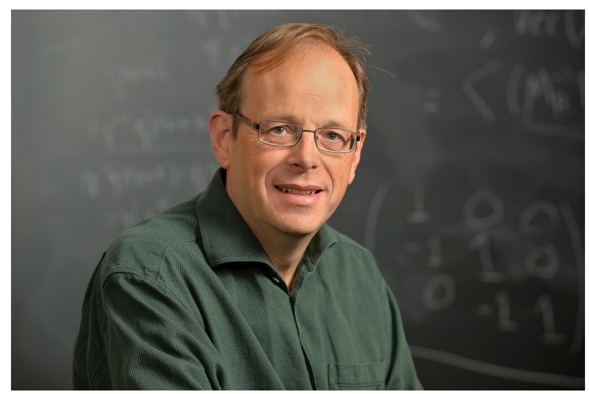
Find integer solutions to $a^n + b^n = c^n$

PROOF DOESN'T FIT HERE

Pierre de Fermat

Find *rational* solutions to $x^n + y^n = 1$

Andrew Wiles



Richard Taylor

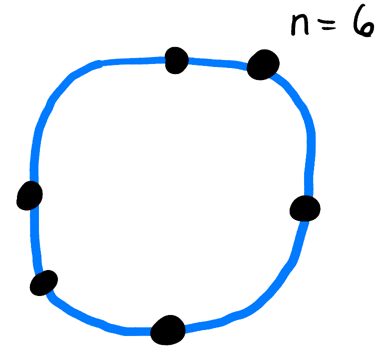
What is algebraic geometry?

We want *rational* solutions to $x^n + y^n = 1$

What is algebraic geometry?

We want *rational* solutions to $x^n + y^n = 1$

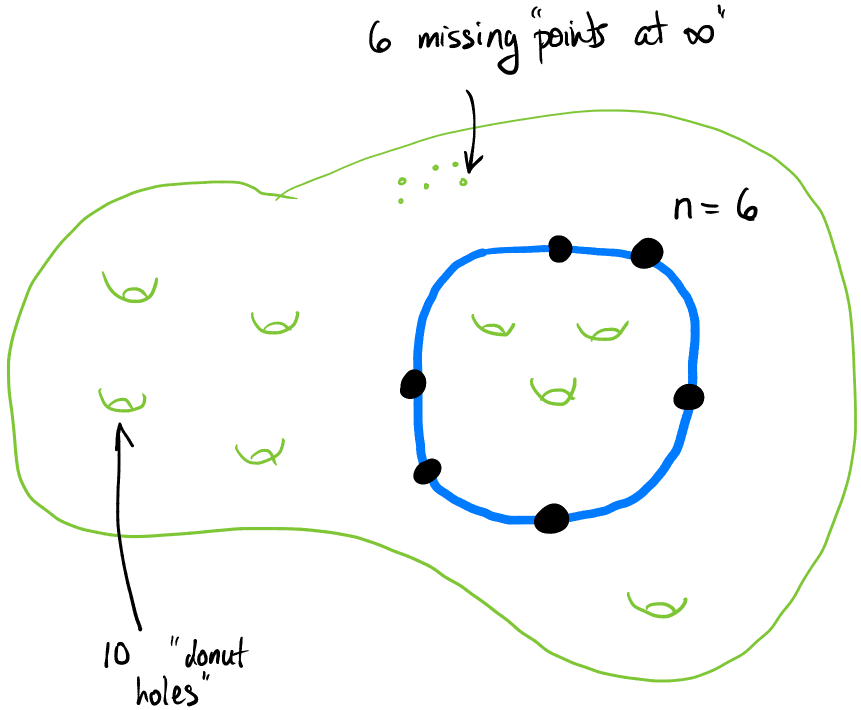
Instead, we picture the
real solutions to $x^n + y^n = 1$



What is algebraic geometry?

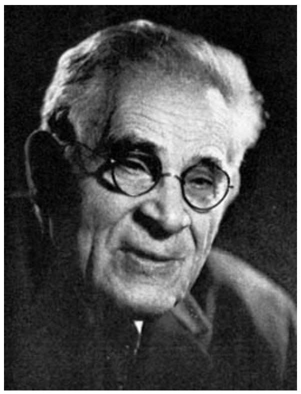
We want *rational* solutions to $x^n + y^n = 1$

Instead, we picture the *real* solutions to $x^n + y^n = 1$



And even the *complex* solutions to $x^n + y^n = 1$

What is algebraic geometry?

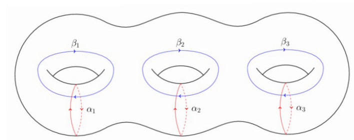


Mordell's Conjecture (Faltings' Theorem) If there are least two "donut holes" in the the *complex* solution set, there can only be finitely many *rational* solutions!

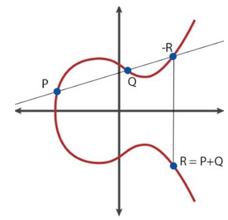
This implies that each case of Fermat's Last Theorem has at most finitely many counterexamples —but it works for all other similar types of problems.

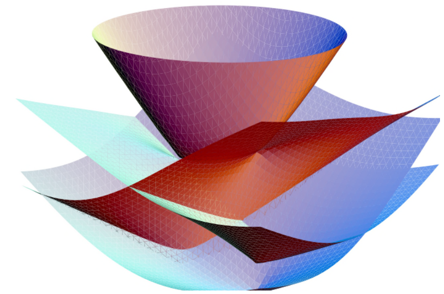
What is algebraic geometry?

complex geometry

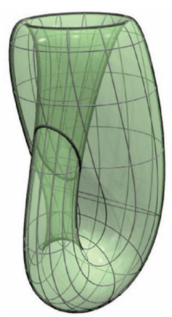


“arithmetic geometry”





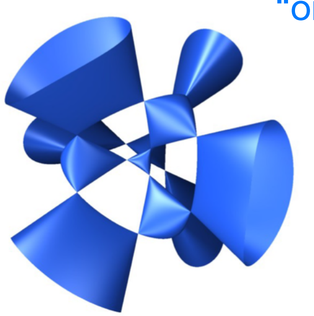
“varieties”



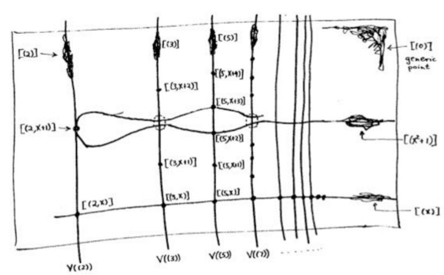
manifolds

The algebra-geometry duality leads to the understanding/invention/discovery of different kinds of “spaces”.

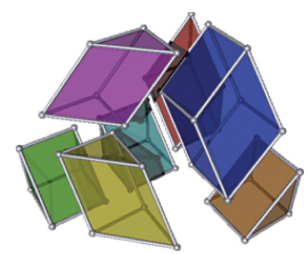
“orbifolds”



“schemes”



“tropical geometry”



What is algebraic geometry?

Caucher
Birkar

Alessio
Figalli

Peter
Scholze

Akshay
Venkatesh

What is algebraic geometry?

"I like crossing the imaginary boundaries people set up between different fields — it's very refreshing."

- Maryam Mirzakhani (1977-2017)

MATHEMATICAL FRONTIERS
Algebraic Geometry

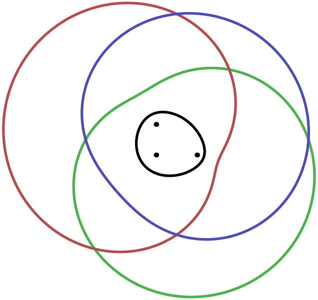
Bernd Sturmfels,
UC Berkeley

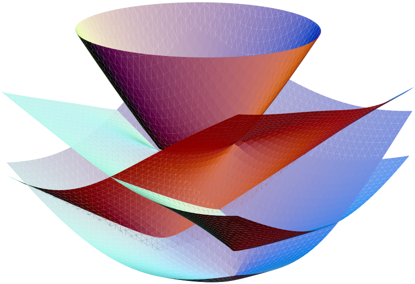
Professor of Mathematics,
Director of the Max Planck Institute for Mathematics in the Sciences in Leipzig, Germany

Algebraic Geometry: Computations and Applications

Algebraic Geometry: Computations and Applications

Bernd Sturmfels
UC Berkeley & MPI Leipzig





Mathematical Frontiers Webinar on **Algebraic Geometry**

May 14, 2019

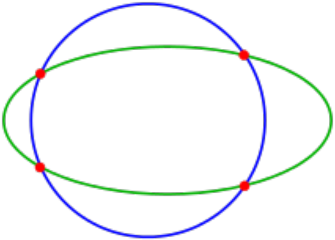
Varieties

A *variety* is the solution set of a system of polynomial equations.

Example: Quadratic curves in the plane:

$$a \cdot x^2 + b \cdot xy + c \cdot y^2 + d \cdot x + e \cdot y + f \quad = \quad 0.$$

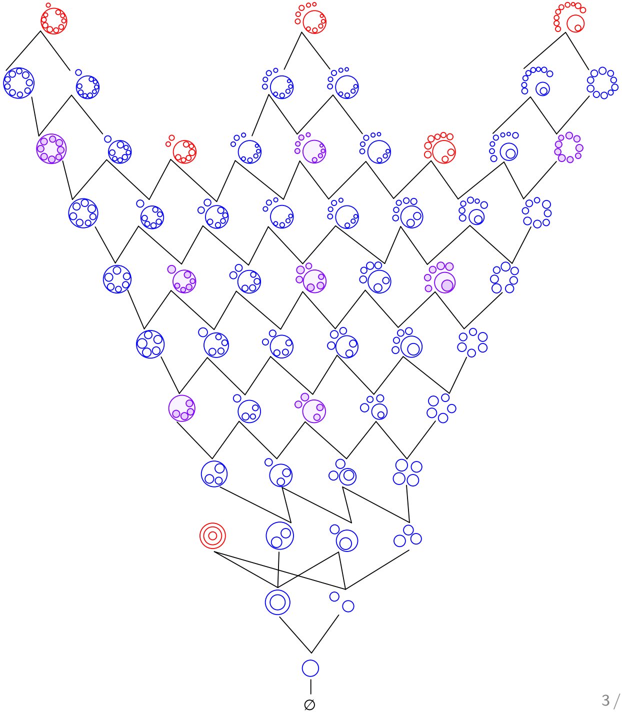
These curves and their intersections are varieties in $\mathbb{R}^2$:



There are almost always four complex solutions. [Bézout 1764]

Case distinction: 0,1,2,3 or 4 real solutions.
The mixed discriminant has 3210 terms in the $12 = 6 + 6$ coefficients.

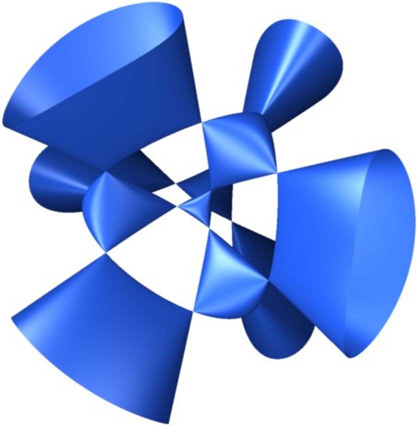
Degree 6 Curves



Kummer Surface

My favorite variety is this surface in 3-space:

$$x^4 + y^4 + z^4 - x^2y^2 - x^2z^2 - y^2z^2 - x^2 - y^2 - z^2 + 1 \quad = \quad 0$$



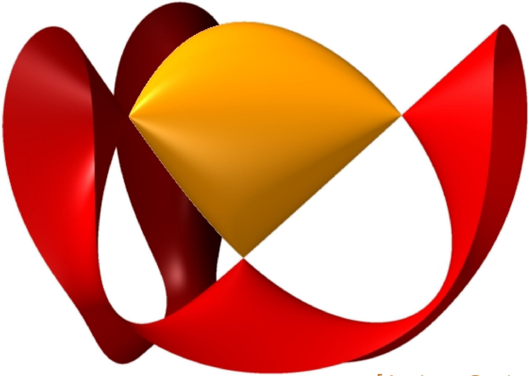
It has 16 singular points. [Ernst Eduard Kummer, 1810-1893]

Kummer surfaces have applications in cryptography.

Convex Optimization

$$x^2 + y^2 + z^2 - 2xyz - 1 \;=\; 0$$

$$\begin{bmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{bmatrix} \succeq 0.$$



[Arthur Cayley, 1821-1895]

Oliver Labs: `surfex.AlgebraicSurface.net`

[Jiawang Nie, Kristian Ranestad, B. St: *The algebraic degree of semidefinite programming*, Math Progr. (2010)]

Semidefinite programming is the foundation for **polynomial optimization**. This has many applications, e.g. in engineering.

Statistics

Watching too much soccer on TV leads to hair loss?

A study asked 296 people about their hair length and how many hours per week they watch soccer on TV. The data:

$$U = \begin{array}{r} \leq 2 \text{ hours} \\ 2\text{–}6 \text{ hours} \\ \geq 6 \text{ hours} \end{array} \begin{pmatrix} \text{full hair} & \text{medium} & \text{little hair} \\ 51 & 45 & 33 \\ 28 & 30 & 29 \\ 15 & 27 & 38 \end{pmatrix}$$

Is there a correlation between watching soccer and hair loss?

Statistics

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Is there a correlation between watching soccer and hair loss?

Extra info: The study involved 126 men and 170 women:

$$U = \begin{pmatrix} 3 & 9 & 15 \\ 4 & 12 & 20 \\ 7 & 21 & 35 \end{pmatrix} + \begin{pmatrix} 48 & 36 & 18 \\ 24 & 18 & 9 \\ 8 & 6 & 3 \end{pmatrix}$$

We cannot reject the **null hypothesis**:

Hair length is conditionally independent of soccer on TV given gender.

Algebraic Statistics

Philosophy: Statistical models are varieties.

Example: *Conditional independence of two ternary variables:*

For a 3×3 data matrix (u_{ij}), one seeks to maximize the likelihood function $p_{11}^{u_{11}} p_{12}^{u_{12}} \cdots p_{33}^{u_{33}}$ over a variety in $\mathbb{R}^9$. It is defined by

$$\det \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \quad = \quad 1 - \sum_{i,j} p_{ij} \quad = \quad 0.$$

This leads to a **system of polynomial equations**. It has almost always 10 complex solutions. How many are real depends on the data.

[J. Hauenstein, J. Rodriguez, B. St: *Maximum likelihood for matrices with rank constraints*, J. Alg. Stat (2014)]
[J. Rodriguez, X. Tang: *Data discriminants of likelihood equations*, ISSAC 2015]

Try This


3264
CONICS
IN A
SECOND
Paul Breiding
Bernd Sturmfels
Sascha Timme
In 1848 Jakob Steiner asked
«How many conics are tangent to five conics?»
In 2019 we ask
«Which conics are tangent to your five conics?»
Curious to know the answer?
Find out at:
juliahomotopycontinuation.org/do-it-yourself/
MAX-PLANCK-GESELLSCHAFT

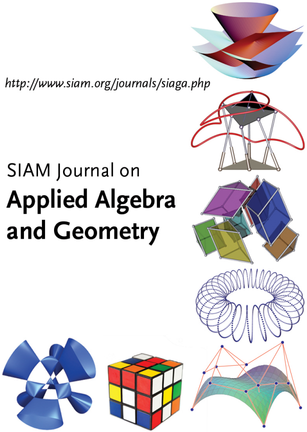
Numerical Algebraic Geometry

Using computers for algebraic geometry started in the 1970's. For 40 years, the emphasis was on **Symbolic Computation**, often via Gröbner bases. Methods are accurate but slow.

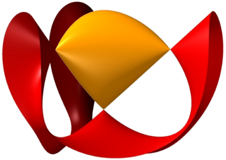
The past decade saw an explosive development of **Numerical Algebraic Geometry**, highlighted by software like Bertini.

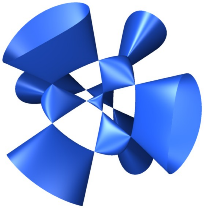
This led to many new applications.

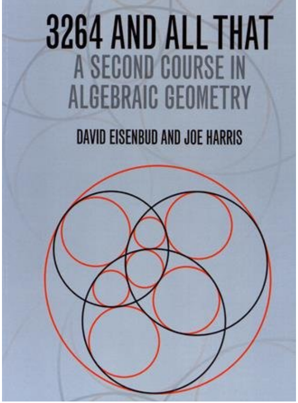
Numerical Software is fast.



Past, Present and Future







$$3264 = 32 \cdot (1 + 5 \cdot 2 + 10 \cdot 4 + 10 \cdot 4 + 5 \cdot 2 + 1)$$

Your five given conics:

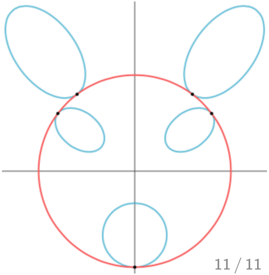
New random conics.

$$0.03x^2 + 0xy + 0.03y^2 + 0x + 0.4y + 1$$
$$2.56x^2 - 2.16xy + 3.19y^2 - 20x - 15y + 75$$
$$2.56x^2 + 2.16xy + 3.19y^2 + 20x - 15y + 75$$
$$22.96x^2 - 19.44xy + 17.29y^2 - 186x - 248y + 2100$$
$$22.96x^2 + 19.44xy + 17.29y^2 + 186x - 248y + 2100$$

Compute tangent conics

3264 complex solutions found in **1.29** seconds.

44 solutions are real: **6** ellipses and **38** hyperbolas.



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