MATHEMATICAL FRONTIERS

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MATHEMATICAL FRONTIERS

2018 Monthly Webinar Series, 2-3pm ET

February 13*:
Mathematics of the Electric Grid

March 13*:
Probability for People and Places

April 10*:
Social and Biological Networks

May 8*:
Mathematics of Redistricting

June 12*: *Number Theory: The Riemann Hypothesis*

July 10*: *Topology*

August 14*: *Algorithms for Threat Detection*

September 11: *Mathematical Analysis*

October 9: *Combinatorics*

November 13:
Why Machine Learning Works

December 11:
Mathematics of Epidemics

*** *Recording posted***

Made possible by support for BMSA from the National Science Foundation Division of Mathematical Sciences and the Department of Energy Advanced Scientific Computing Research

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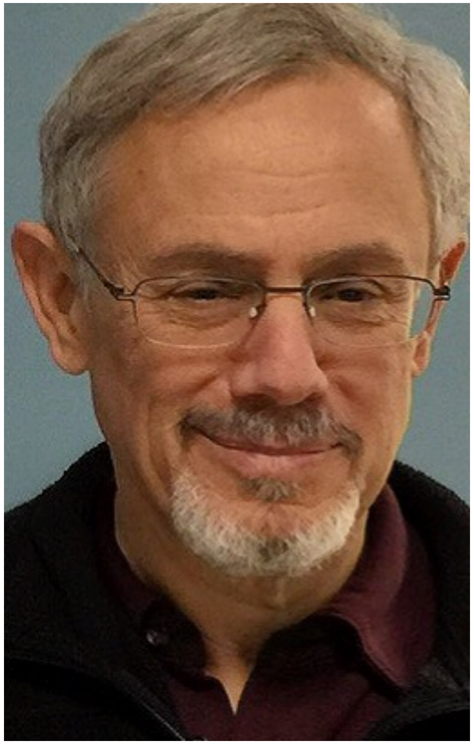
MATHEMATICAL FRONTIERS

Mathematical Analysis

Dimitri Shlyakhtenko,
UCLA

Svitlana Mayboroda,
University of Minnesota



Mark Green,
UCLA (moderator)

MATHEMATICAL FRONTIERS
Mathematical Analysis

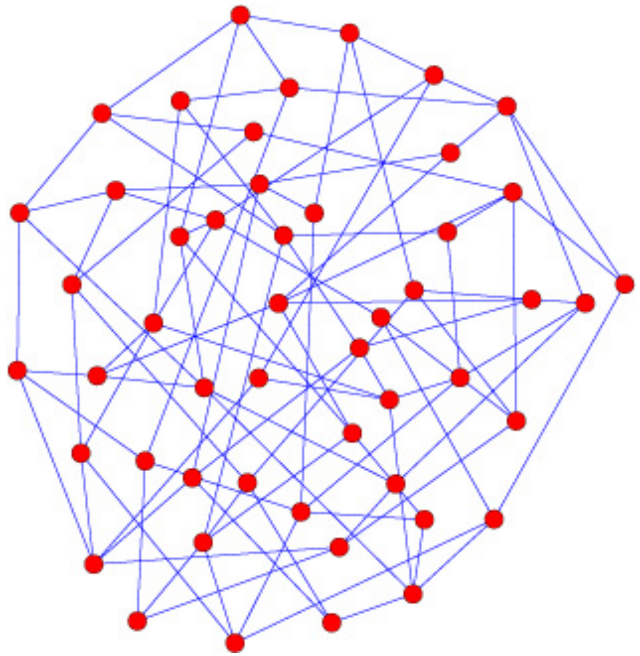
Dimitri Shlyakhtenko,
UCLA

Director of Institute for Pure and Applied Mathematics and
Professor of Mathematics in the Department of Mathematics at the University of California, Los Angeles

Marcus, Spielman, Srivastava: Making Sparse Graphs

Graphs

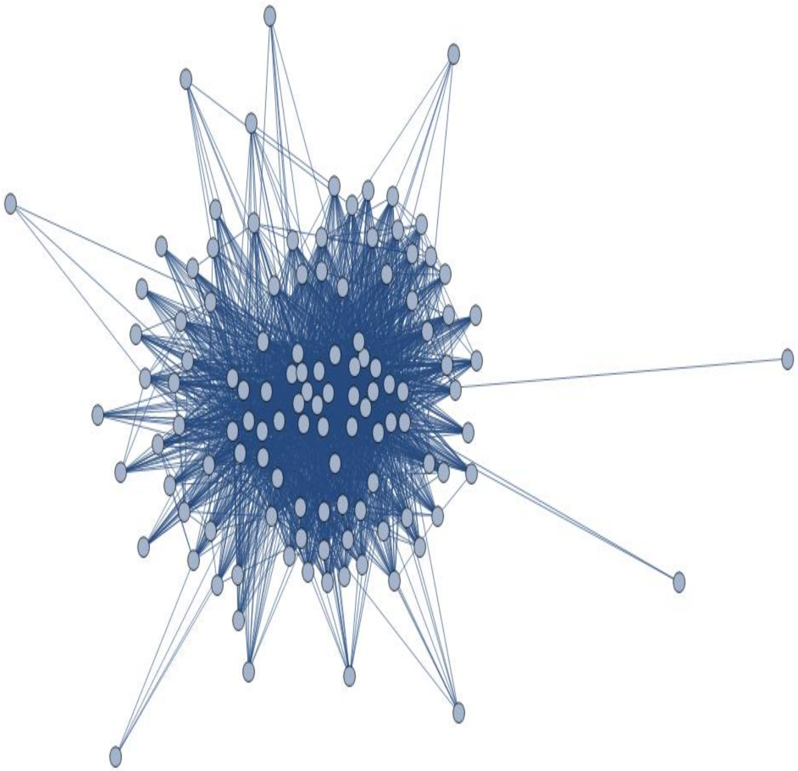
- A *graph* consists of a set of vertices *V* and a set of edges *E* between them:



Graphs are useful to encode relationships:
- Vertices = people, edges = shared interest
- Vertices = Senators, edges = same vote on an issue
- Networks: vertices = computers, edges = connections

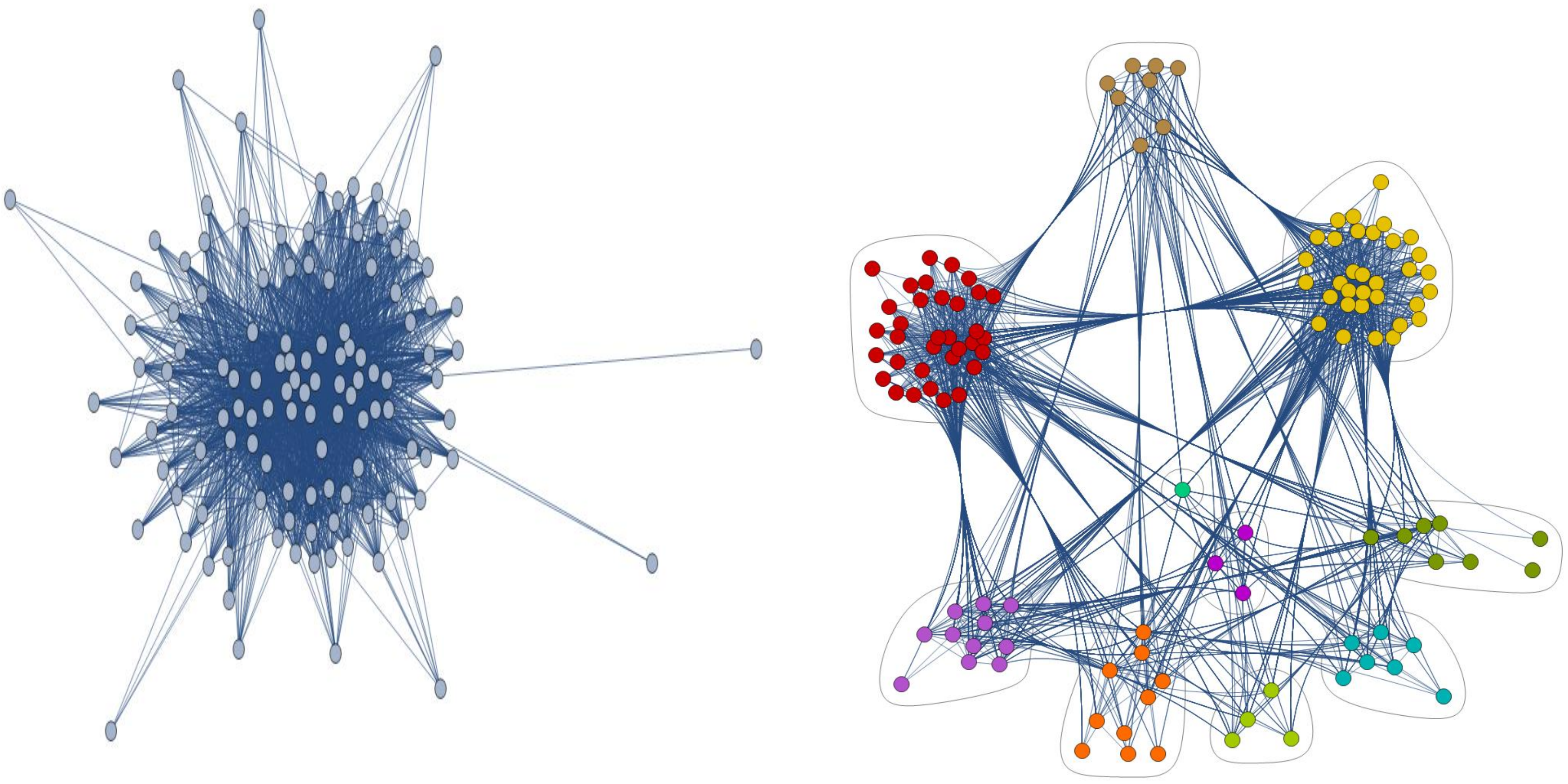
Typical questions:
  - Clustering / community detection
  - Cuts / reliability / speed of info

Community detection:
119 IPAM Programs (last 20 years)



Vertices = programs, Edges = common participants

Community detection: 119 IPAM Programs (last 20 years)

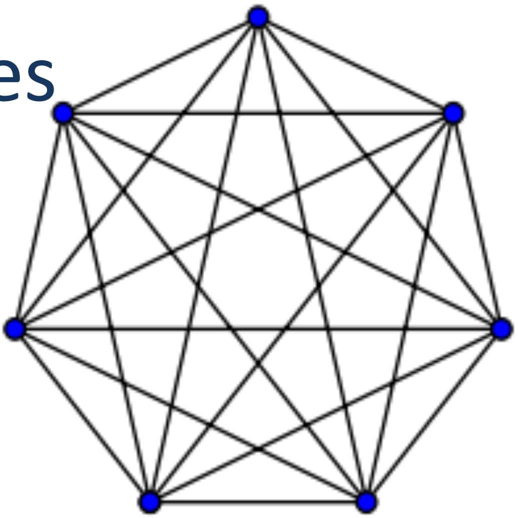


Vertices = programs, Edges = common participants

Problem

- Too many edges!
- E.g. Complete Graph (all pair of vertices connected)

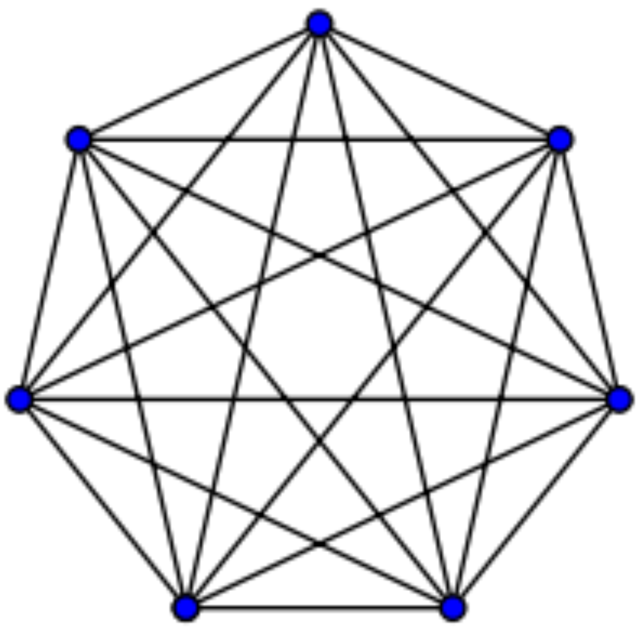
  N vertices gives $N(N-1)/2$ edges



- Very well connected, but at great cost: connecting 1,000 computers pairwise requires ~500,000 wires!

Sparsity

- Goal: replace a complicated graph by a graph with much fewer edges, while keeping the same "essential features".
- Compute with the sparse graph (much easier!) to detect communities, find cuts, etc.
- Replace "all to all" complete graph by a graph with much fewer edges but similar properties.
- Amazing progress in this area: Benczur-Karger'96, Spielman-Teng'04, Spielman-Srivastava'08, Marcus-Spielman-Srivastava'14,…



Complete graph

“Similar” explained

- One measure of similarity is the speed at which heat would spread across the graph if a single vertex is heated.

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“Similar” explained

- One measure of similarity is the speed at which heat would spread across the graph if a single vertex is heated.
- Graphs with high heat spread rates are called *expanders.*
- A complete graph is an expander.
- **Challenge**: find graphs with relatively few edges which are expanders (“sparsify complete graphs”).

Expanders.

- Expanders are very useful in many areas of mathematics, computer science, etc.
  - “sparsification” of complete graph
  - Data correction (“expander codes”)
  - Cryptography (Hash functions)
  - …..

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- Expanders are very useful in many areas of mathematics, computer science, etc.
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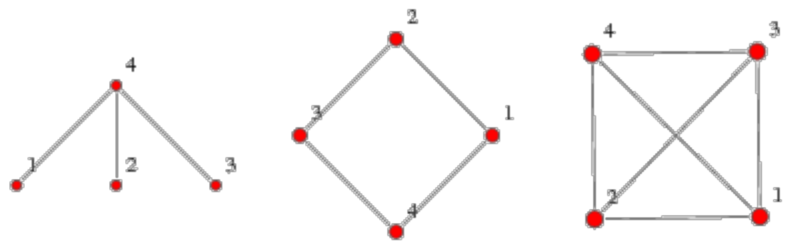
  - Cryptography (Hash functions)
  - …..
- Challenge: construct good expanders

From Graphs to Matrices

- Each graph can be encoded by its *adjacency matrix A.*
- The matrix has an entry for each pair of vertices, with 1 or 0 depending on whether they are connected.

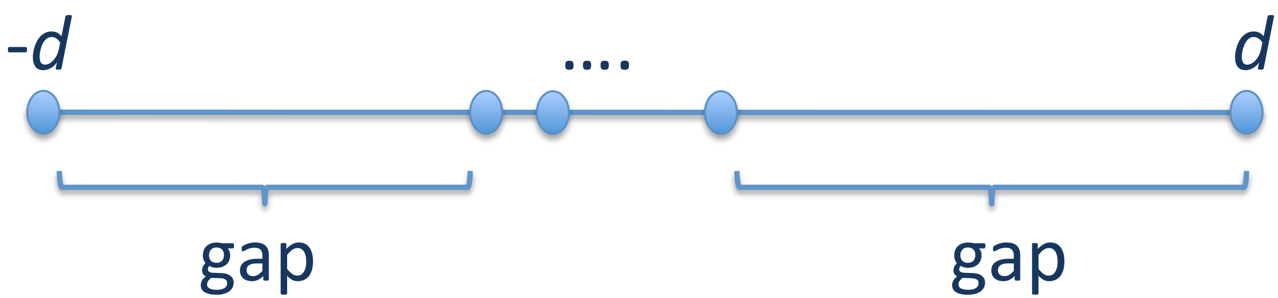


$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Credit: WolframAlpha

Expanders via matrix eigenvalues

- Assume each vertex has d edges ("degree d"). Compute *eigenvalues* of the matrix $L=d\mathbf{I}-A$.
- The eigenvalues of L are numbers in the interval $[-d,d]$:



Size of the two gaps is related to heat dissipation rate (how good of an expander the graph is).

How to build expanders

- Goal: construct graphs with fixed degree *d* (=number of edges at each vertex) but which are good expanders (big gap).

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How to build expanders

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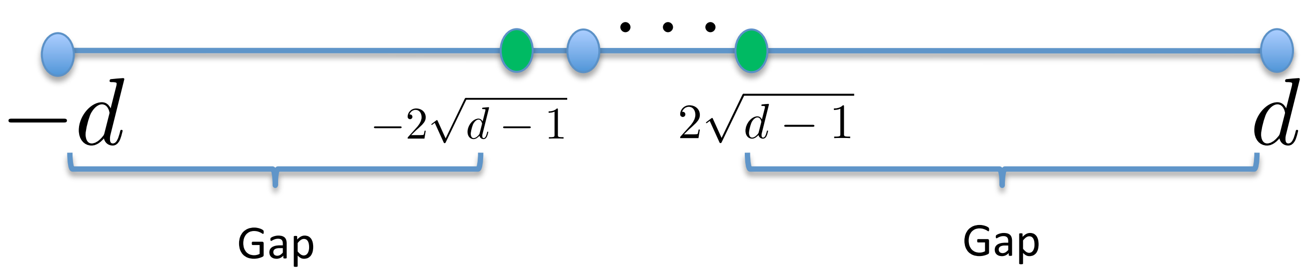
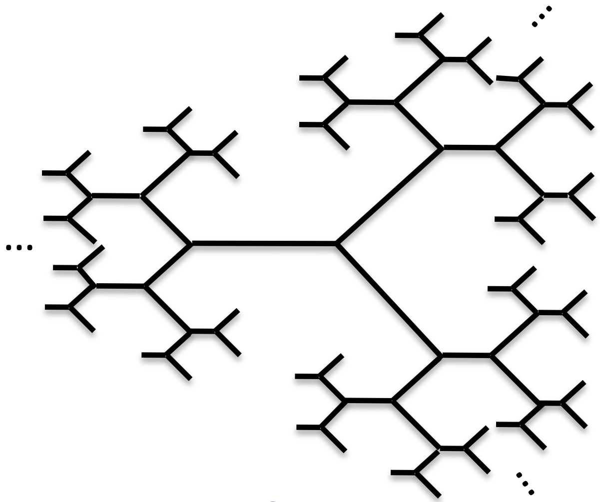


Alon-Boppana bound

- The gap bound is actually realized by an *infinite tree*.

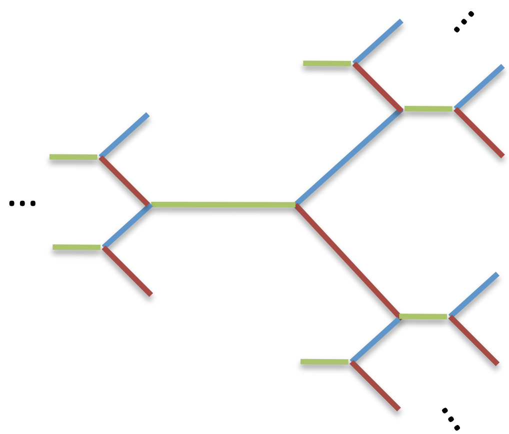


IPAM 2018 program "Quantitative Linear Algebra": connections between finite and infinite matrices.

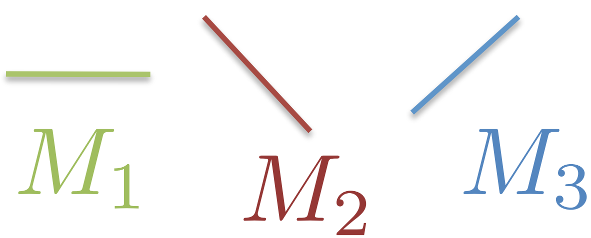
- But we want a *finite* graph with *d* edges at each vertex reaching the bound. These are called **Ramanujan graphs**. Finite is harder than infinite!

Trees

- What are trees made of?



- A tree is a *free product* of *matchings.*



Trees

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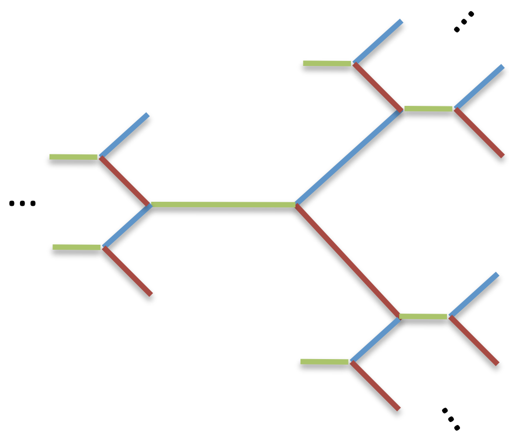
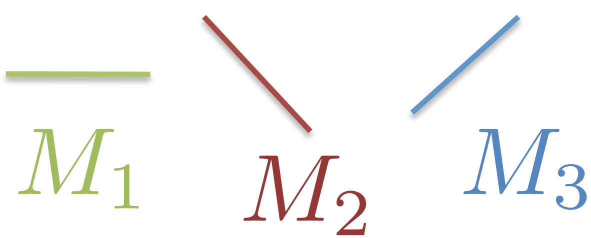


$L_{tree} = M_1 + M_2 + M_3$

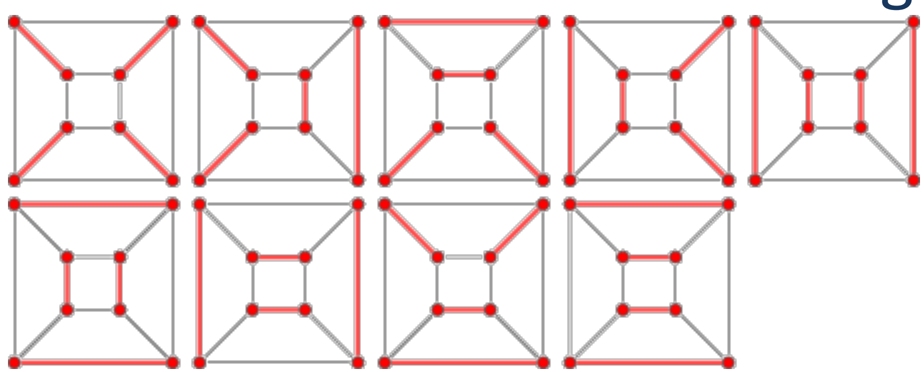
Where M_1, M_2, M_3 are *freely independent* matchings.

- A tree is a *free product* of *matchings.*



From infinite to finite

- So one should try to build a Ramanujan graph as a free product … but free products always give *infinite* graphs.
- Idea: free independence (Voiculescu) occurs for random matrices when their sizes grow.
- Take $2N$ points and randomly match N of them to the rest. Do this d times to create d matchings $Q_1,\ldots,Q_d$



Matchings, $2N=8$ *Credit: WolframAlpha*

Construction of Ramanujan Graphs

- **Theorem** [Marcus, Spielman, Srivastava'14] With non-zero probability the sum of the random matchings $Q=Q_1 + \ldots + Q_d$ is the matrix of a Ramanujan graph.

Construction of Ramanujan Graphs

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In fact, this can be done constructively giving examples of Ramanujan graphs with an arbitrary fixed number of edges per vertex. [Lubotsky-Phillips-Sarnak'86 only for $d=p+1$]

Further directions

- Finding Ramanujan graphs is related to the question of how to sparsify a complete graph.
- Further techniques of Marcus-Spielman-Srivastava can be used to sparsify other graphs.
- → Better algorithms for graph analysis!
- In math, better connections between finite and infinite dimensions.

Further Reading

- Lecture series by Srivastava at the Simons Institute: https://simons.berkeley.edu/talks/nikhil-srivastava-2014-08-27
- Tutorials at IPAM, including lectures by Srivastava and Tao: https://www.ipam.ucla.edu/programs/workshops/quantitative-linear-algebra-tutorials/?tab=schedule

MATHEMATICAL FRONTIERS

Mathematical Analysis

Svitlana Mayboroda,
University of Minnesota

Director of Simons Collaboration on Localization of Waves and Northrop Professor of Mathematics in the School of Mathematics at the University of Minnesota

Localization of Waves

Localization of Waves

The results presented in this talk are a part of a newly established Simons collaboration "Localization of Waves" including the PIs

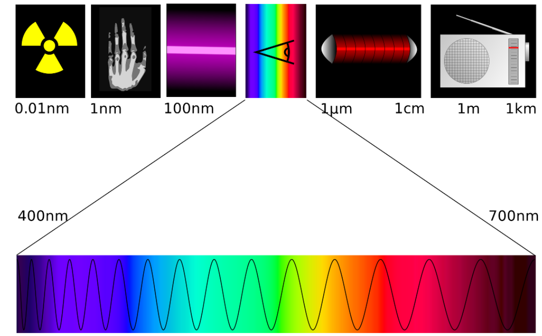
- D. Arnold, UMN (applied math)
- A. Aspect, Institut d'Optique (cold atoms)
- G. David, Université Paris-Sud (harmonic analysis, geometric measure theory)
- M. Filoche, Ecole Polytechnique (condensed matter physics)
- R. Friend, Cambridge (organic semiconductors)
- D. Jerison, MIT (harmonic analysis, PDE)
- Y. Meyer, ENS-Cachan (harmonic analysis)
- J. Speck, UCSB (GaN semiconductors)
- C. Weisbuch, UCSB and Ecole Polytechnique (semiconductors)

and our many collaborators.

A world full of waves

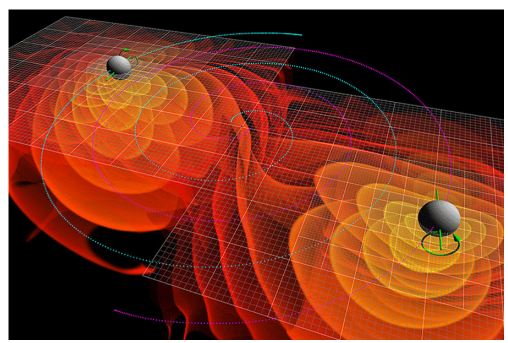
Mechanical

Fluid

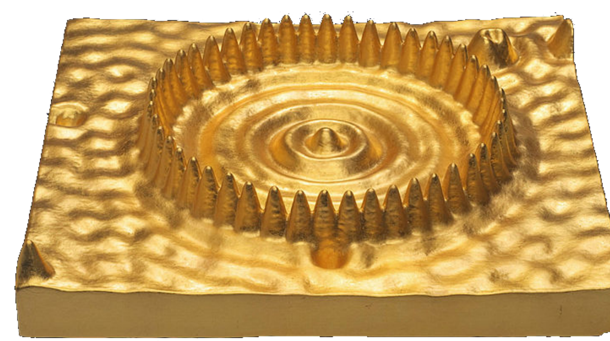


Electromagnetic

Acoustic



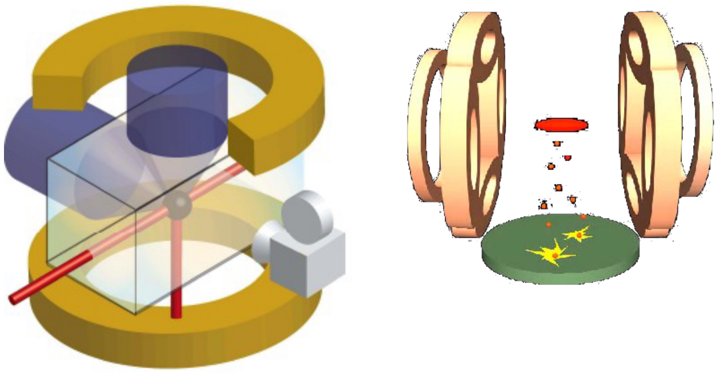
Gravitational



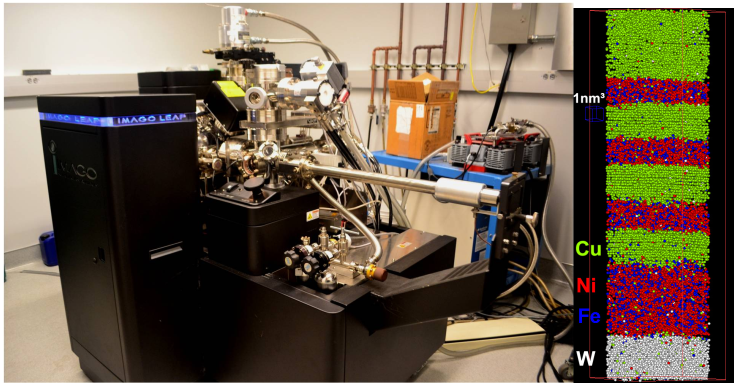
Matter

Add technology which makes the invisible visible…

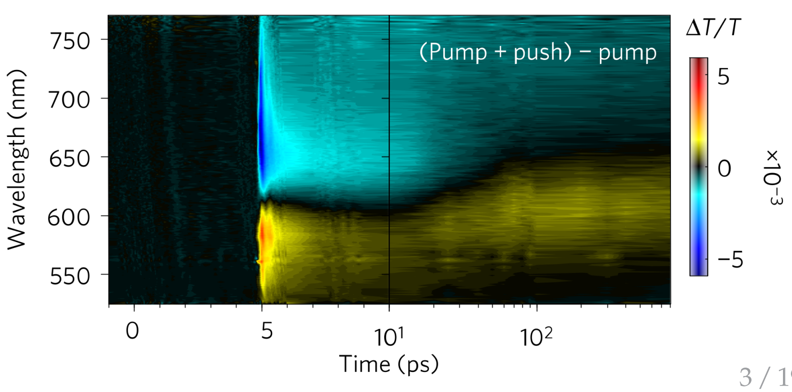
- Manipulate individual atoms with laser cooling: a few billionths of a degree above absolute zero
  [Aspect lab, Institut d'Optique]



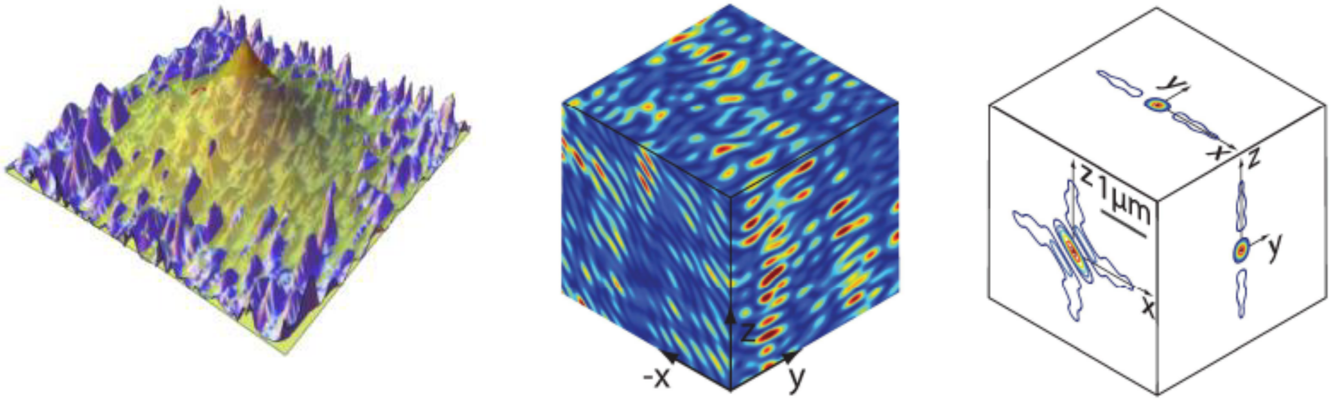
- Map out materials atom-by-atom: samples of 10,000-10,000,000 nm^3 with billions of atoms
  [Speck & Weisbuch lab, UCSB]



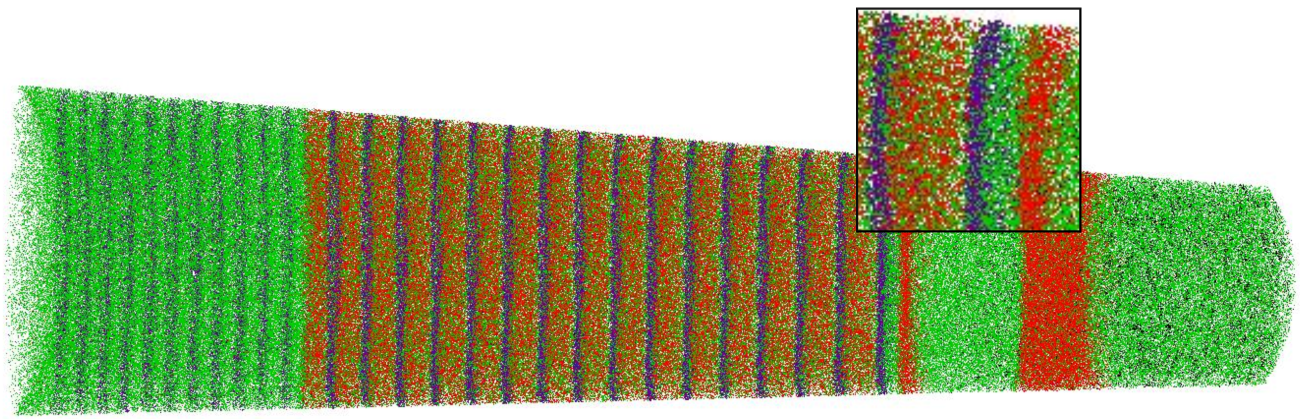
- Pump-push-probe electroabsorption: 100 fs $= 10^{-13}$ s migration of holes from higher to lower energy sites
  [Friend lab, Cambridge]



And we find disorder everywhere

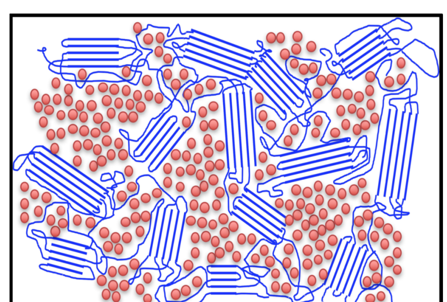


Anderson localization in Bose-Einstein condensate [Aspect lab]



Atomic map of InGaN semiconductor [Speck lab]

Mixed donor-acceptor morphology in an organic solar cell [Friend lab]



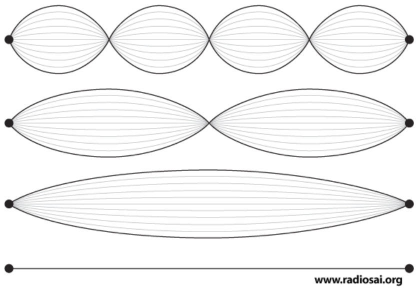
String vibration

Any string vibration is a linear combination of "harmonics" – eigenfunctions which solve

$$\begin{cases} -\frac{d^2}{dx^2}\varphi = \lambda\varphi, \\ \varphi(0) = \varphi(1) = 0 \end{cases}$$

λ are the corresponding eigenvalues (AKA energies)

Here we assume that the string is fixed at the ends and has length 1: $u(0) = u(1) = 0$

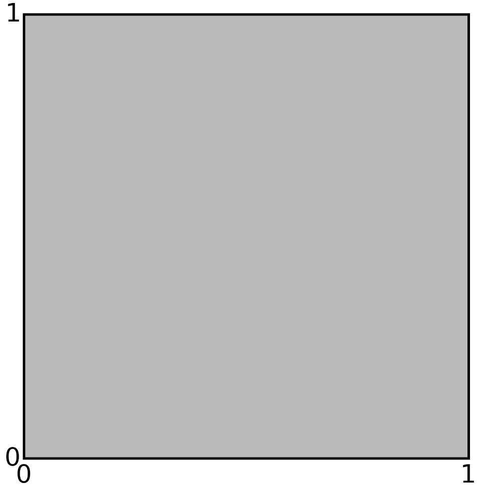


$$\varphi(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$$

Given $\varphi(0) = \varphi(1) = 0$, we have

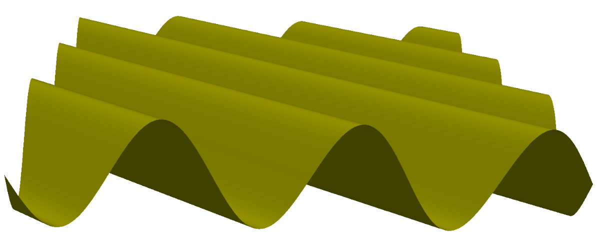
$$\lambda_n = (n\pi)^2, \quad \varphi_n = \sin(n\pi x)$$

Smooth versus disordered potential in Schrödinger equation



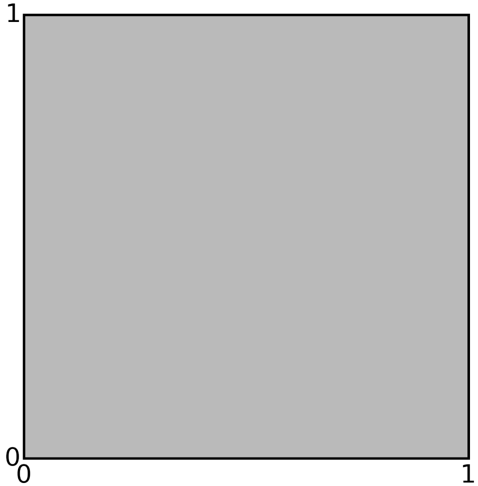
no potential

fundamental mode

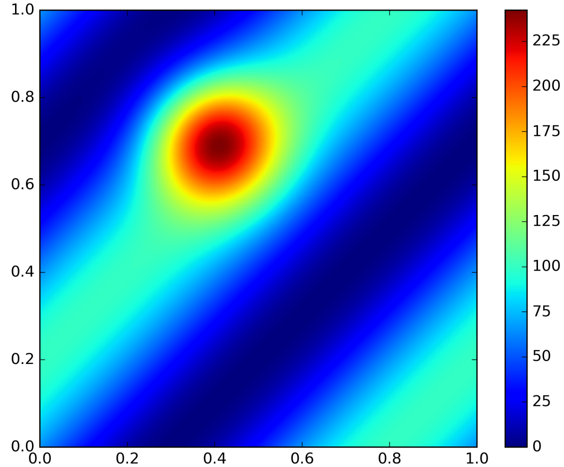


57th mode

Smooth versus disordered potential in Schrödinger equation

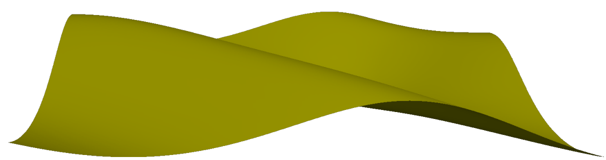


no potential



smooth potential

fundamental mode



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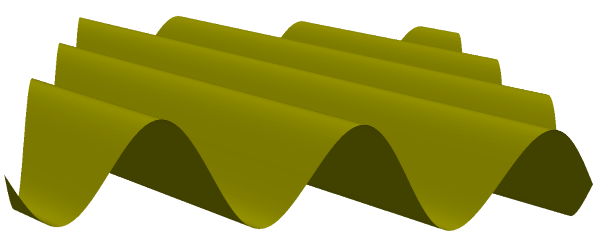

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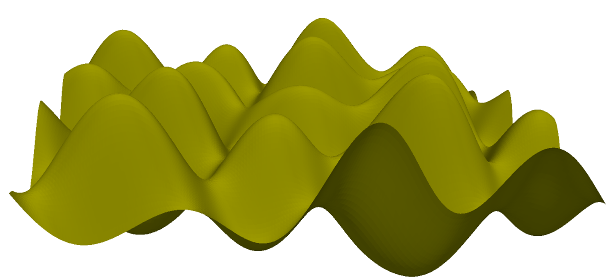

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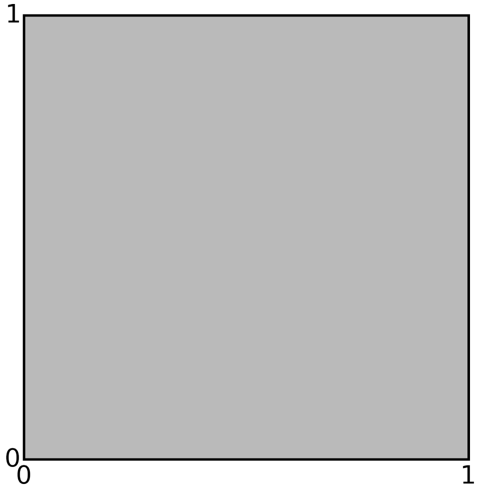
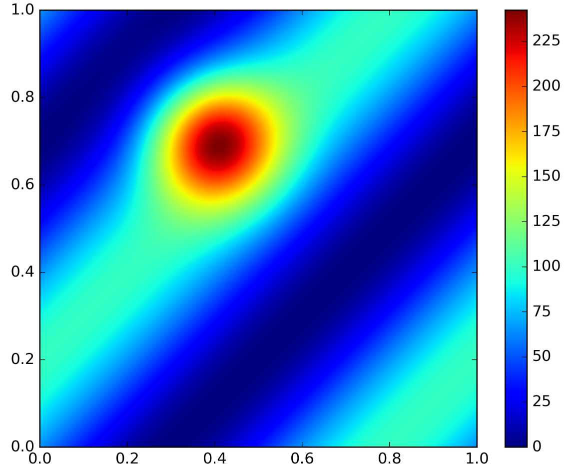
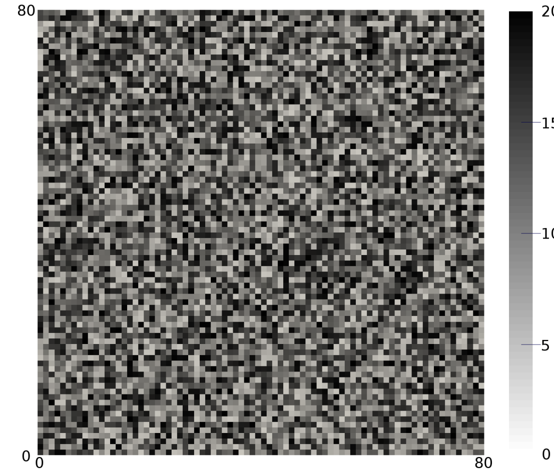


no potential



smooth potential



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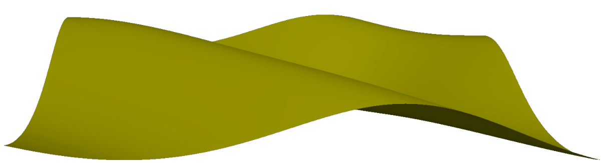

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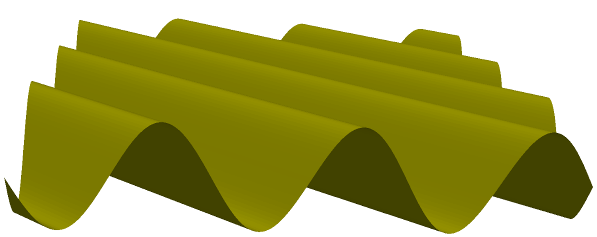

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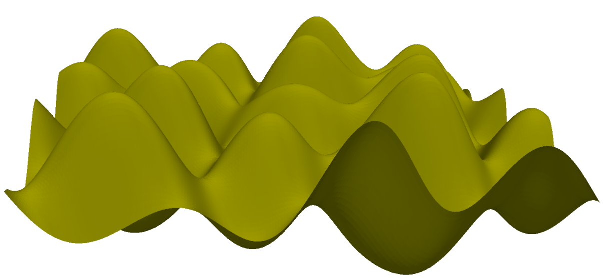

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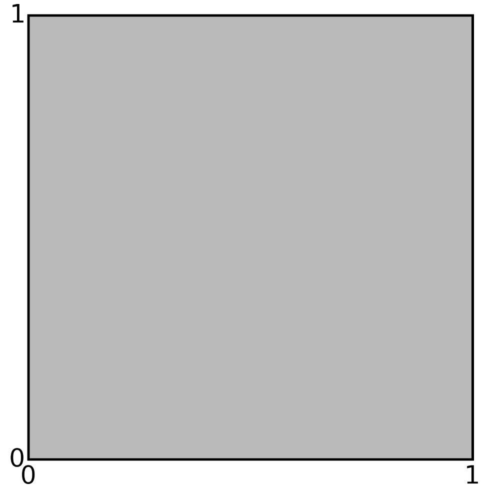
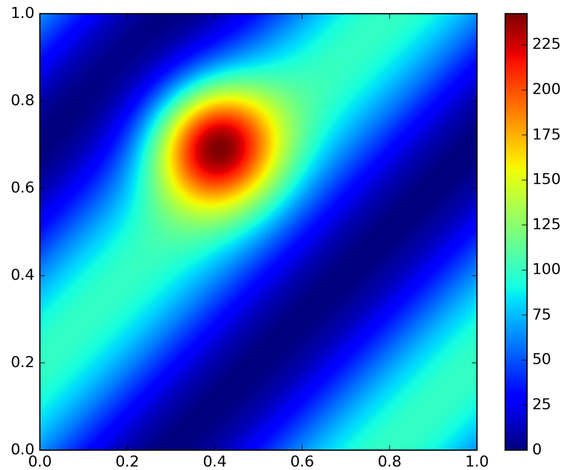
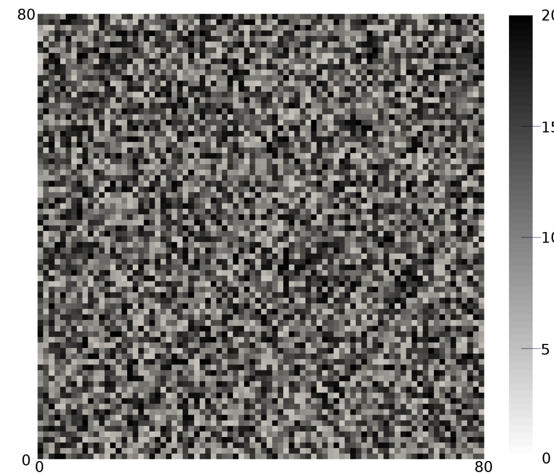


no potential



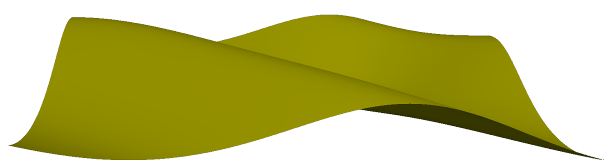
smooth potential



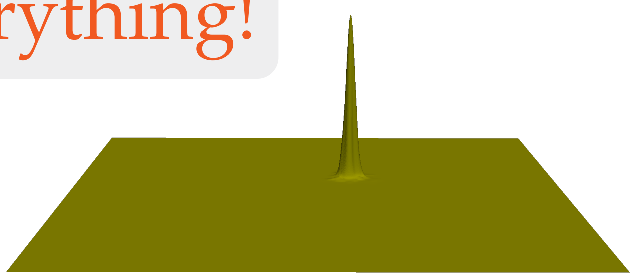
random potential

Disorder changes everything!

fundamental mode



fundamental mode



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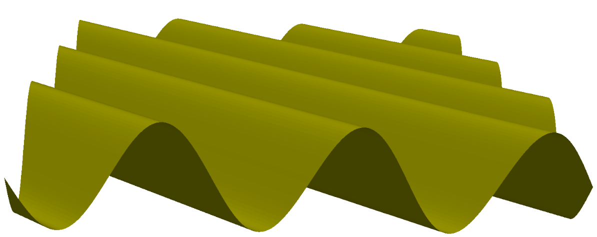

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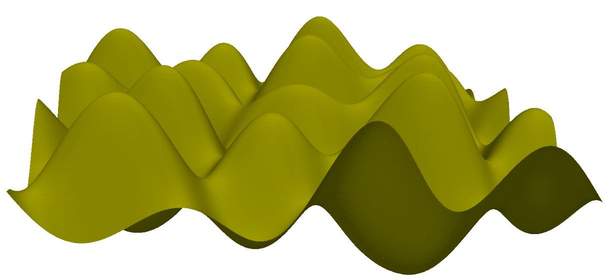


57th mode

57th mode

The main goal: basic questions

We seek a quantitive, deterministic understanding of the mechanisms of wave localization so we can answer such questions as:

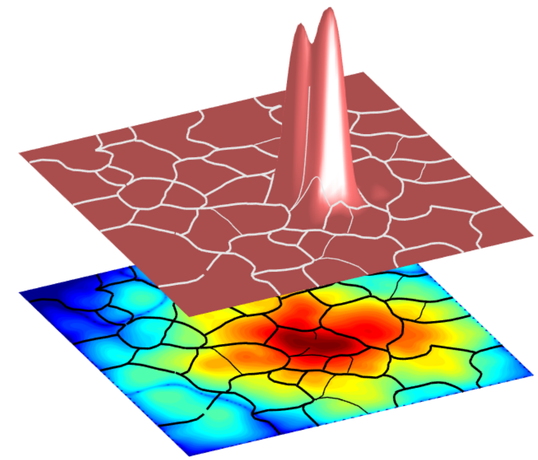
- When and where do eigenfunctions localize?
- How many localize?
- What are the size and shapes of their supports?
- What are the associated eigenvalues?

More generally, for Schrödinger and far more complex systems, we want to:

- Determine wave behavior in a given disordered environment.
- Infer the disordered environment by observing wave behavior.
- Design the environment to obtain desired or optimal wave behavior.

Take on the perspective of a wave

The main hero:
THE LANDSCAPE



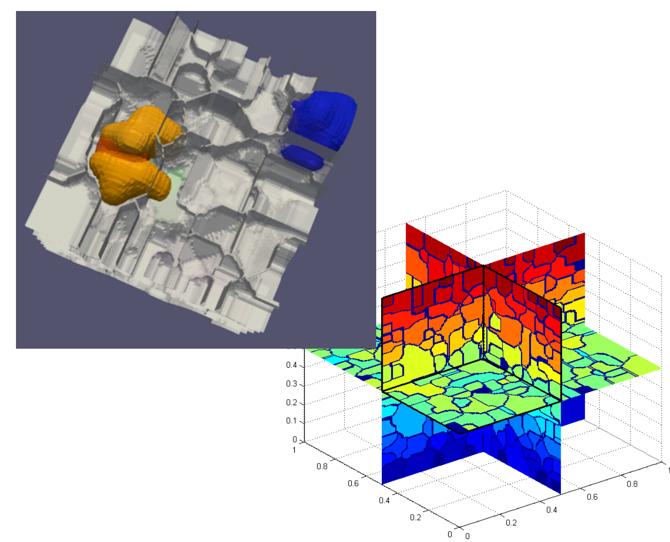
A hidden landscape that waves recognize and obey

- born of the equation but invisible to the naked eye
- contains both spatial and spectral information

The goal is to

Discover and master this landscape in order to

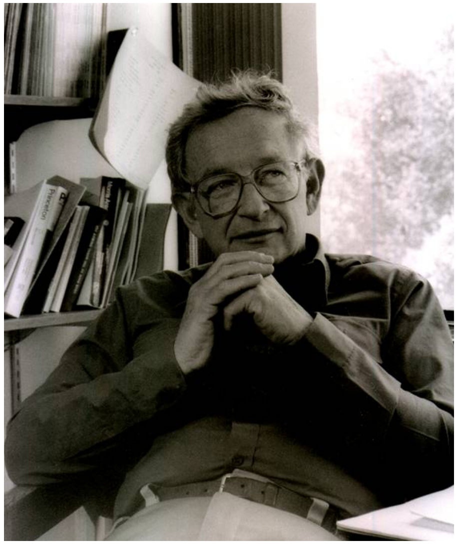
- understand
- predict
- manipulate
- govern
- and, ultimately, design matter waves



Curves/surfaces of the landscape vs. eigenfunctions

Anderson localization

The localization of Schrödinger eigenfunctions with random potential was discovered by Philip Anderson in his Nobel-prize-winning work of 1958.

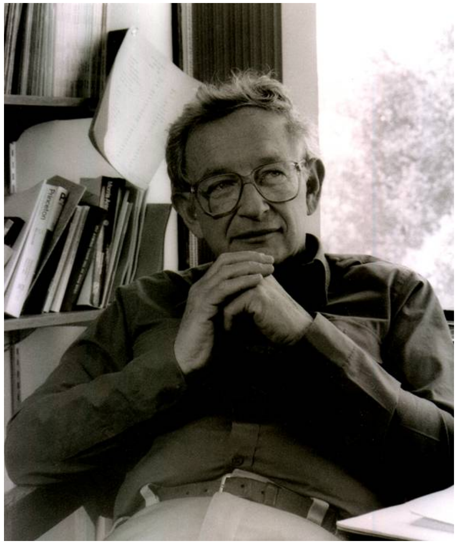


> *Unfortunately, electron localization was devilishly hard to confirm... experimental observations are sparse and covered with disputes and controversies.*
>
> – Lagendijk, van Tiggelen, Wiersma, *50 Years of Anderson Localization*, 2009

> *Most theoretical work [7-9] predicts [the critical exponent]* $\mu = 1$*, but there is also a prediction of* $\mu = 1/2$ *[10]. Numerical simulation [11] gives* $\mu = 2/3 \ldots$
>
> – I. Shlimak, *Is Hopping a Science?*, 2015

Anderson localization

The localization of Schrödinger eigenfunctions with random potential was discovered by Philip Anderson in his Nobel-prize-winning work of 1958.



> *Very few believed [localization] at the time, and even fewer saw its importance; among those who failed to fully understand it at first was certainly its author. It has yet to receive adequate mathematical treatment, and one has to resort to the indignity of numerical simulations to settle even the simplest questions about it.*
>
> – Philip W. Anderson, *Nobel Lecture*, 1977

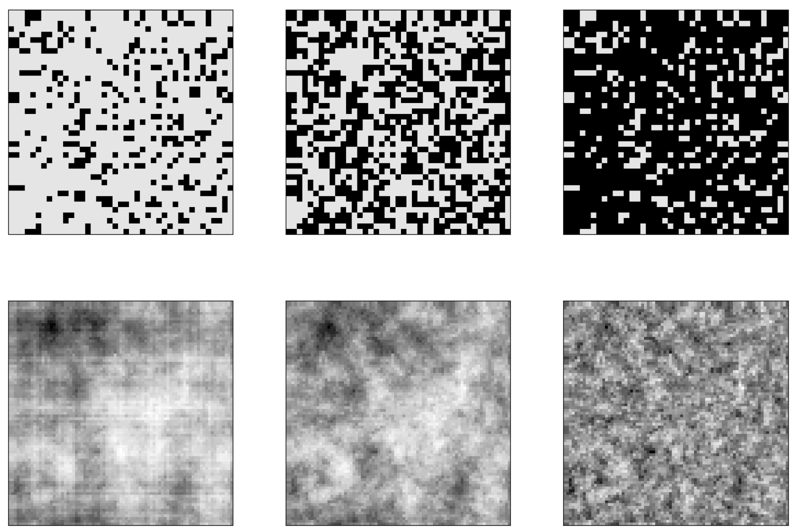
Localization beyond Anderson

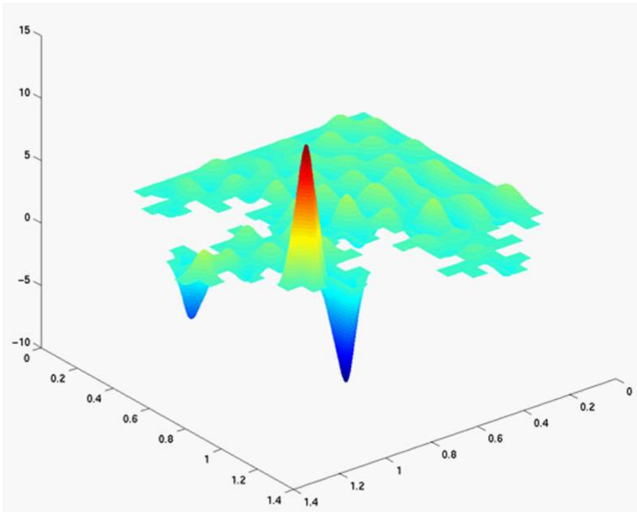
Localization occurs in many, many situations of interest.

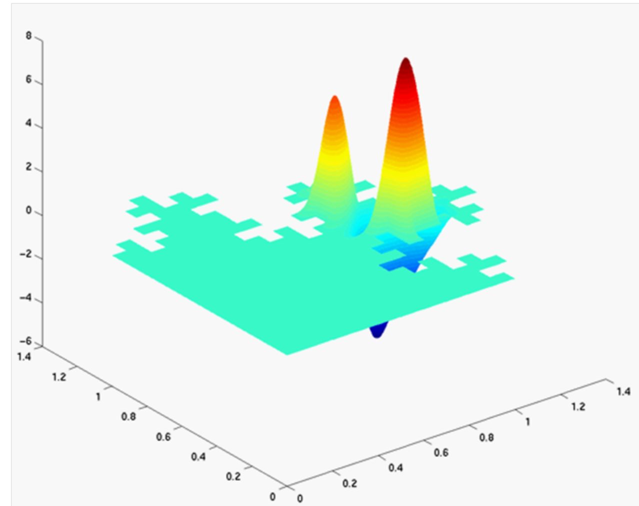
- localization in 2D, diffusion in 3D, higher dimensions
- correlations
- singular distributions (e.g., Bernoulli)
- deterministic disorder (almost Mathieu, quantum Hall effect)
- localization by geometry (e.g., Bernoulli with values 0 and $+\infty$)

Bernoulli potential

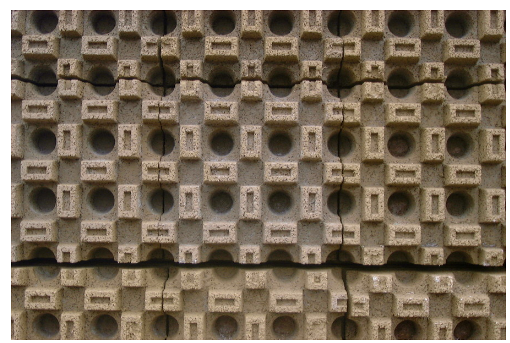
Correlated Gaussian field potential







High-order localized Dirichlet modes of a 2D domain with fractal-like boundary



Fractal® Wall Acoustic Barrier, Filoche et al.

Localization in mathematics

A collection of rather different phenomena:

- semiclassical analysis
- mathematical physics
- probability
- PDEs
- geometric measure theory
- harmonic analysis

... to mention only a few

A new tool for ordering the disorder

The landscape function

Theorem (Filoche & Mayboroda 2012)

For H be any elliptic operator (possibly with rough coefficients, complex boundary, ...), define the landscape function u by the equation

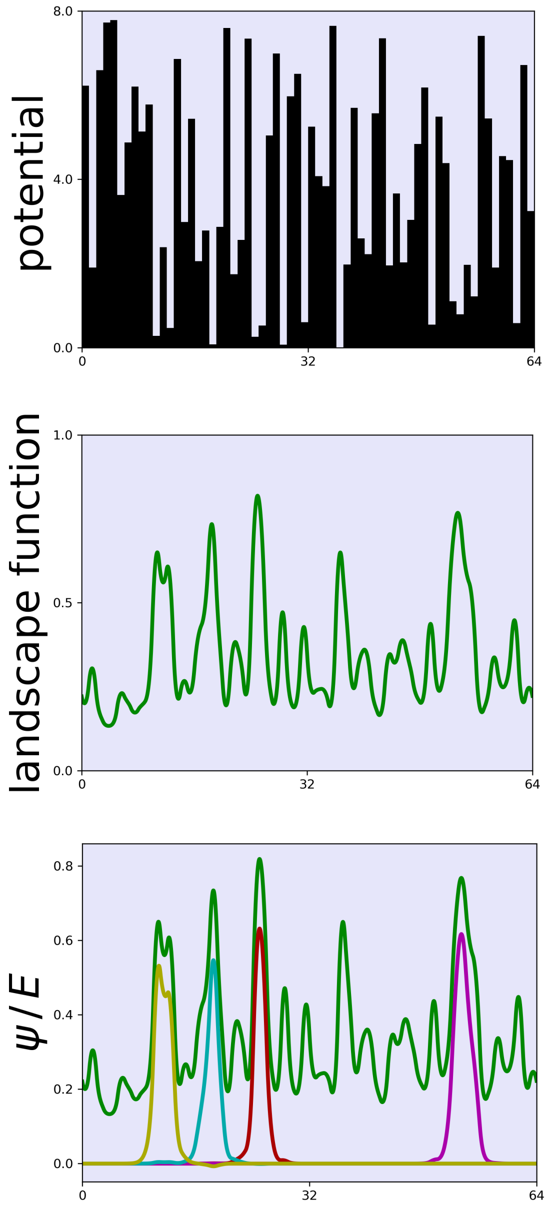
$$Hu = 1$$

plus boundary conditions. Then, any normalized eigenpair (E, ψ),

$$H\psi = E\psi, \quad \|\psi\|_{L^\infty} = 1,$$

satisfies the pointwise bound

$$|\psi(x)| \leq E\, u(x).$$



A different perspective: the effective potential

Arnold, David, Filoche, Jerison, Mayboroda, 2015–2018:

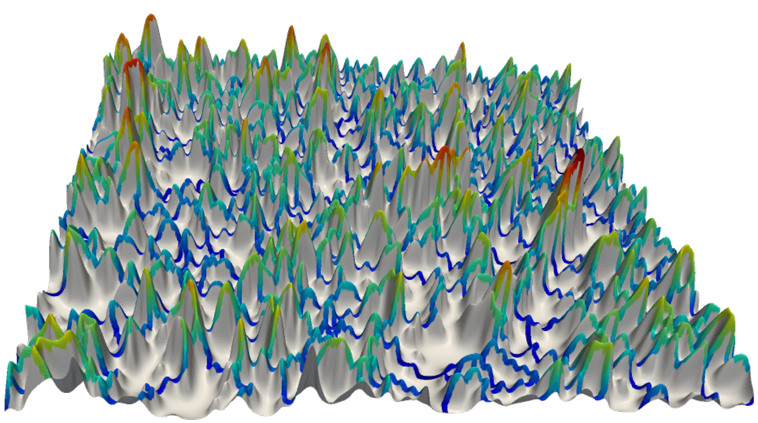
linear equation $\Longrightarrow$ nonlinear control

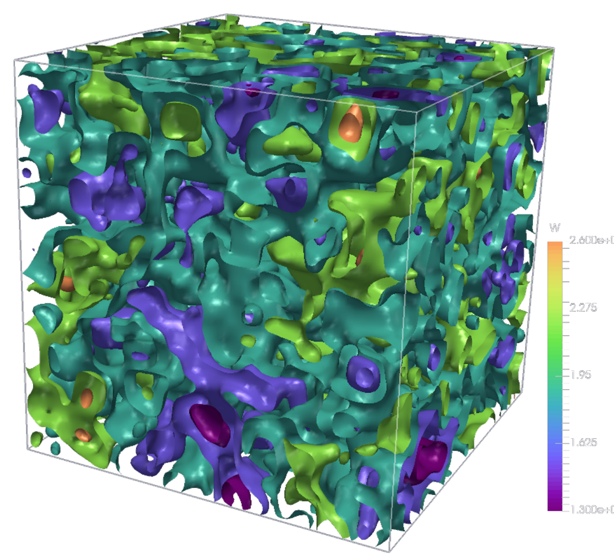
$W = 1/u$ is an *effective potential* which is often confining. The new eq.

$$-\frac{1}{u^2}\nabla\cdot(u^2\nabla\phi) + \frac{1}{u}\phi = E\phi$$

has exactly the same eigenvalues as the Schrödinger equation.

$Hu = 1 \quad \Longrightarrow \quad$ enhanced *Agmon-type distance* $\rho_{1/u} \quad \Longrightarrow \quad$ exp decay

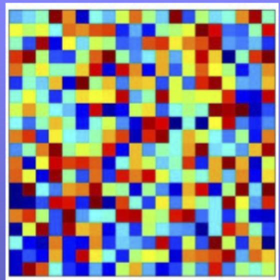




2D and 3D effective potential $\frac{1}{u}$ for Bernoulli V

Localization by disorder: $L = -\Delta + V$

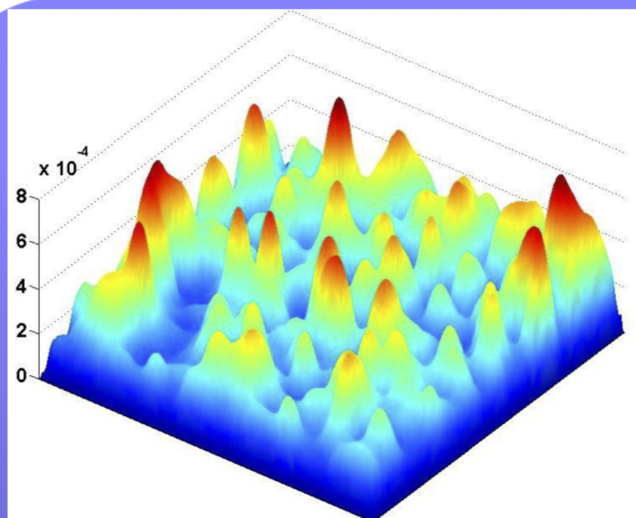
Operator *L*



The localization scheme

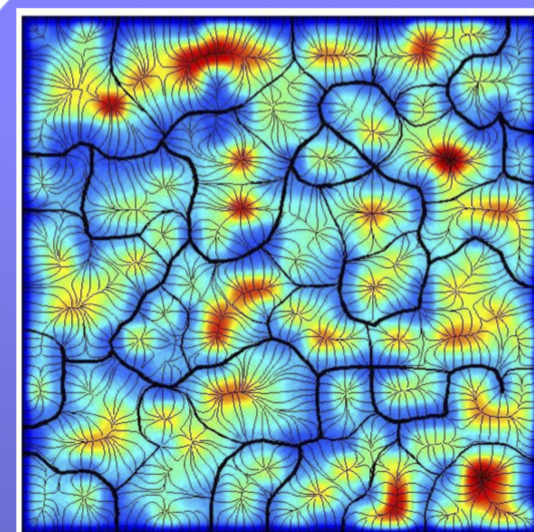
Quantum states in a random potential

$$L\psi(x,y) = E\psi(x,y)$$



Main new idea: **the landscape** *u*

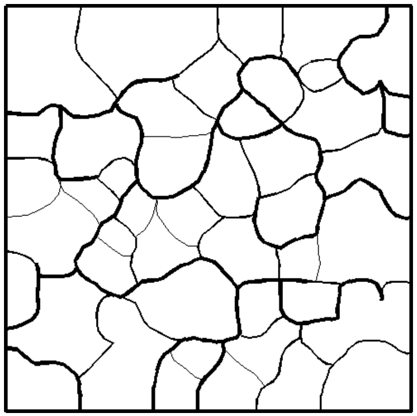
$$L\,u(x,y) = 1$$



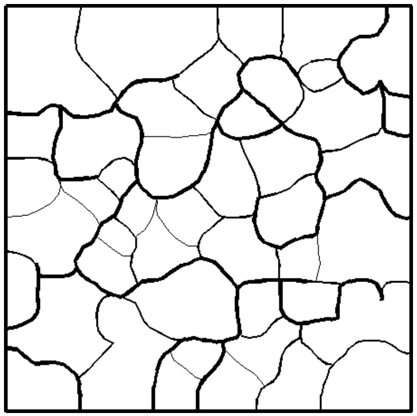
Valley network of *u*
(black curves)

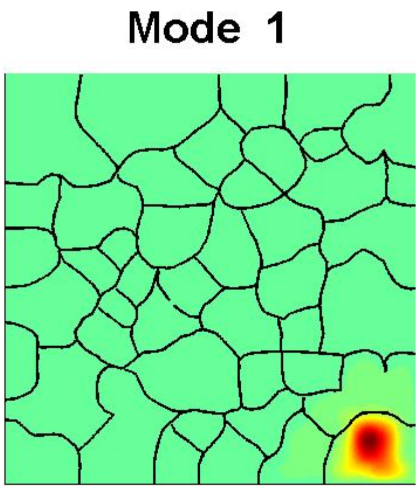
Filoche & Mayboroda, PNAS, 2012

Anderson Localization: valleys and quantum states

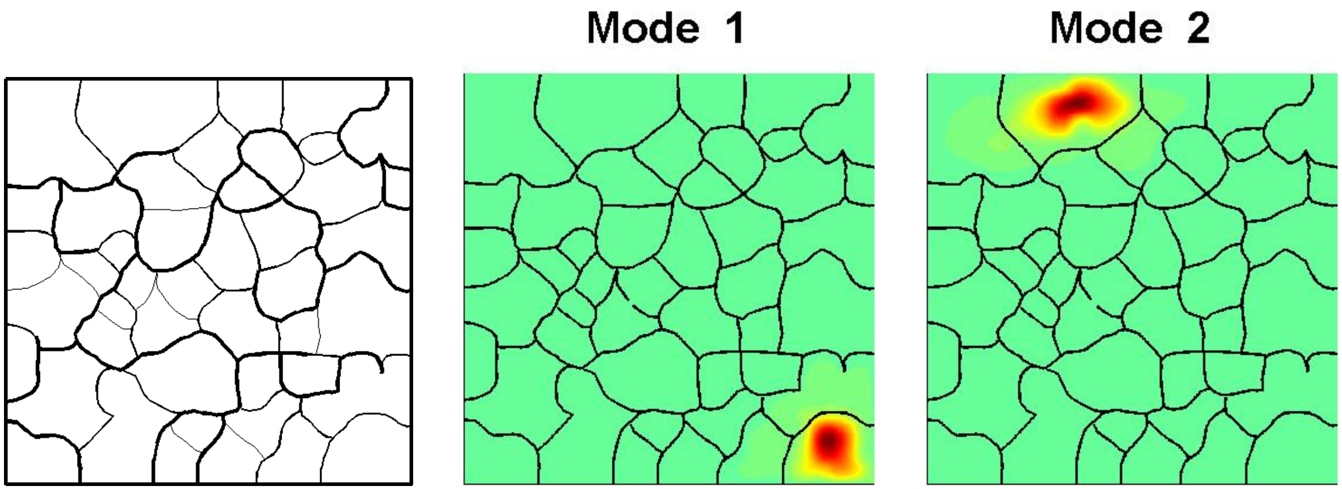


Anderson Localization: valleys and quantum states

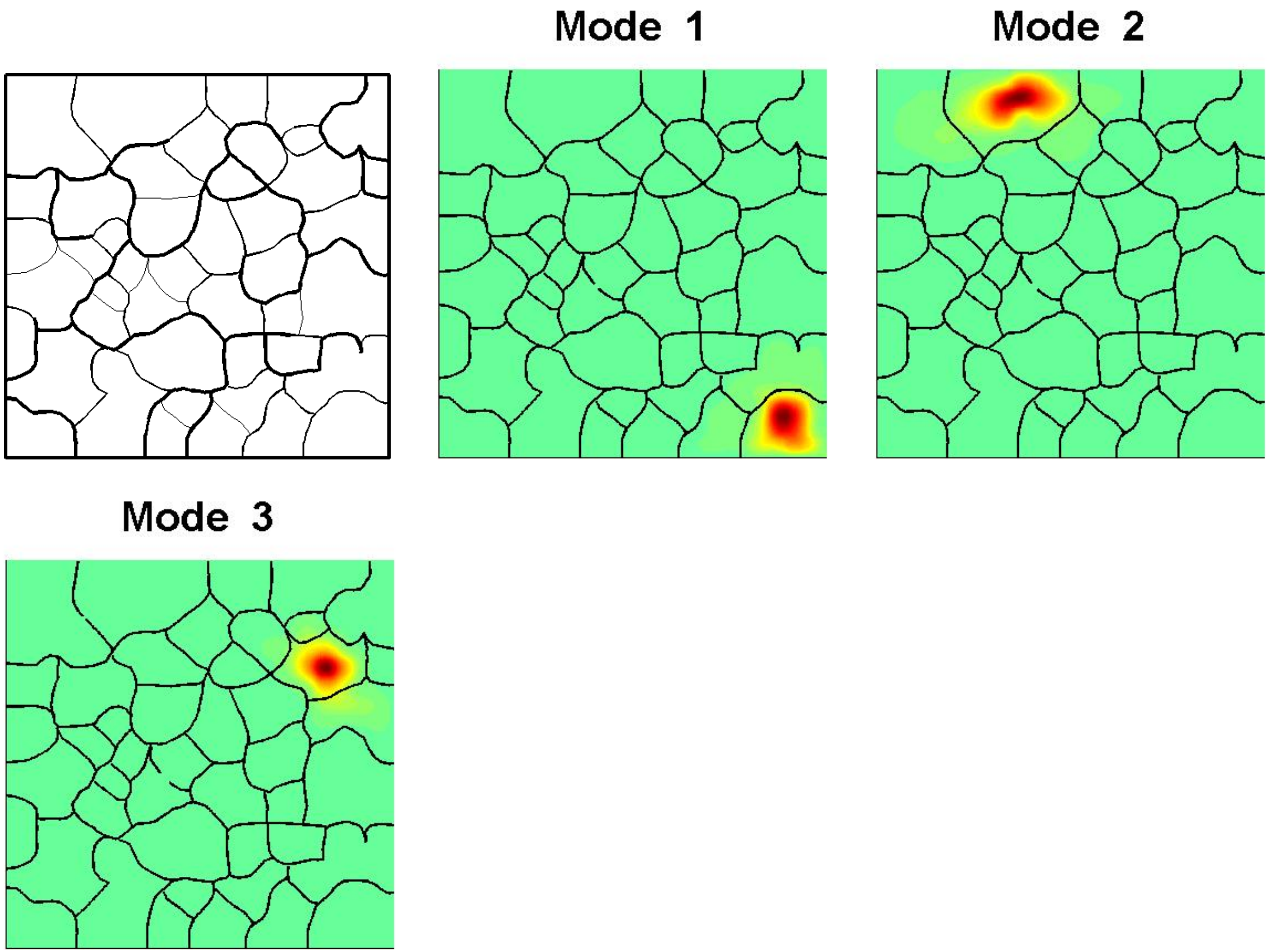




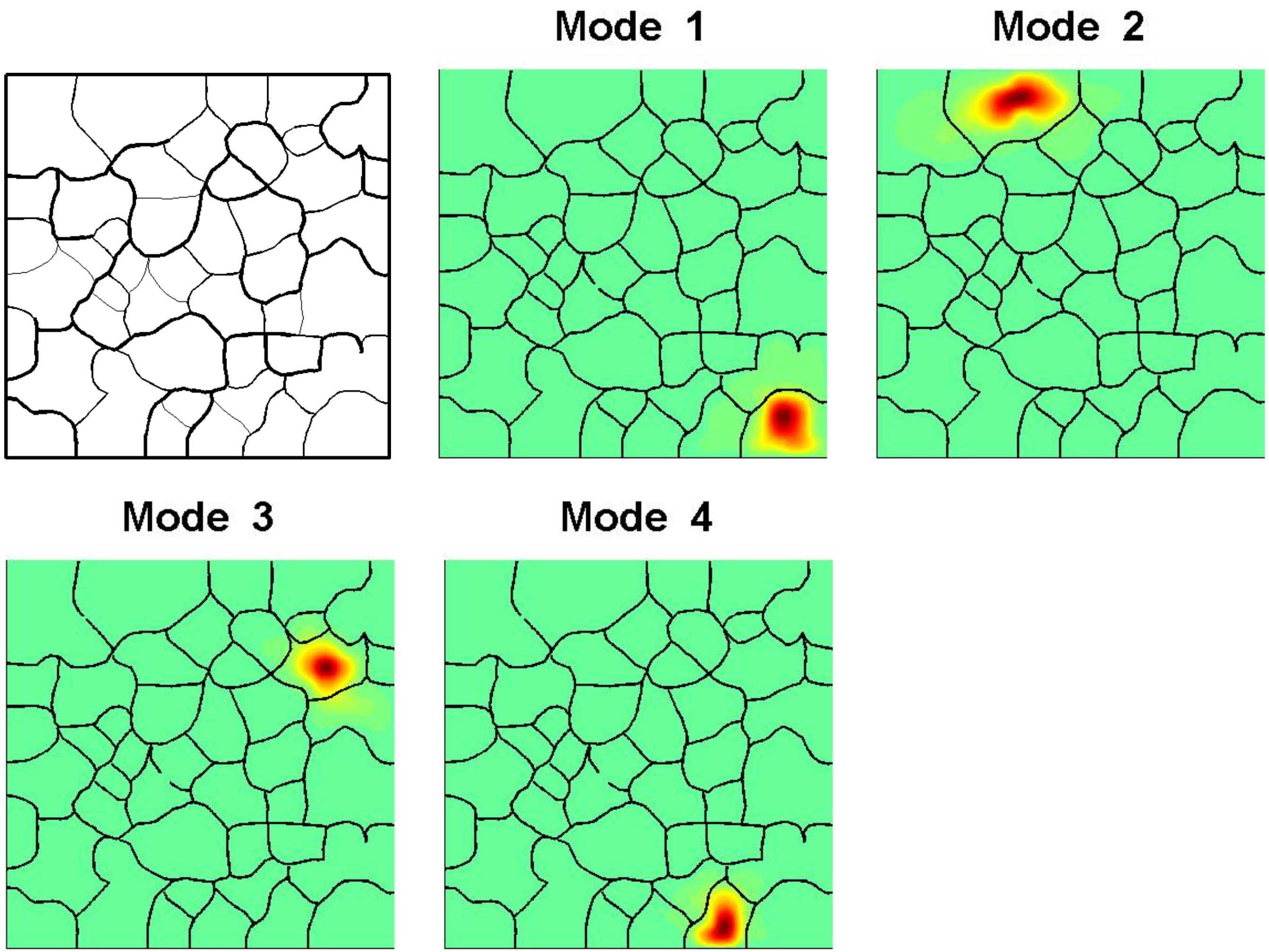
Anderson Localization: valleys and quantum states



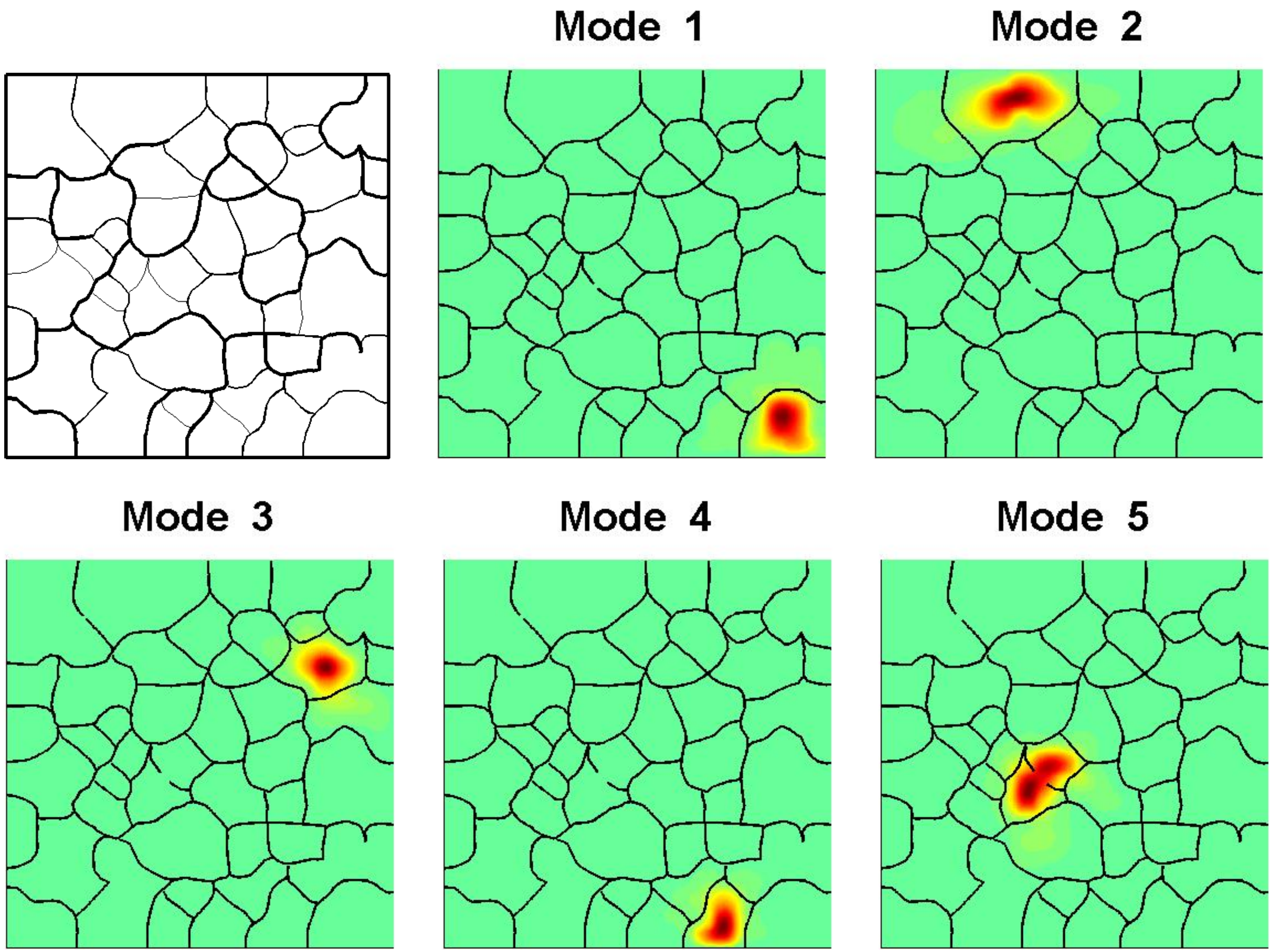
Anderson Localization: valleys and quantum states



Anderson Localization: valleys and quantum states



Anderson Localization: valleys and quantum states



What does the localization landscape reveal?

$1/u$ is an effective potential
(joint with Arnold, David, Filoche, Jerison)

- The watershed basins associated to its minima predict the localization regions.

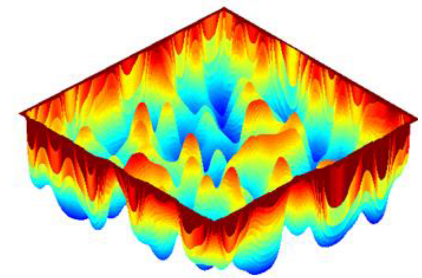
$$\frac{\psi(x)}{\max \psi} \le E u(x)$$

- Its crests induce exponential decay.

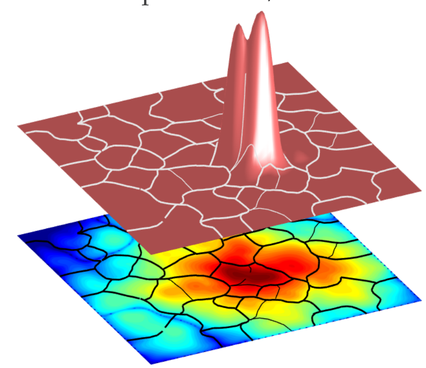
$$\psi(x) \sim \exp\left\{-\rho_{\left(\frac{1}{u}-E\right)_+}(x)\right\}$$

- Its well depths predict the energy levels.

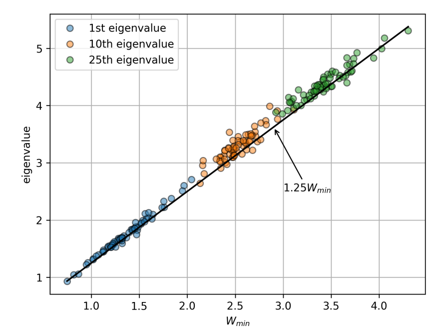
$$\min \frac{1}{u} \approx \left(1+\frac{n}{4}\right) E$$



effective potential $1/u$



watershed lines of $1/u$ and an eigenfunction

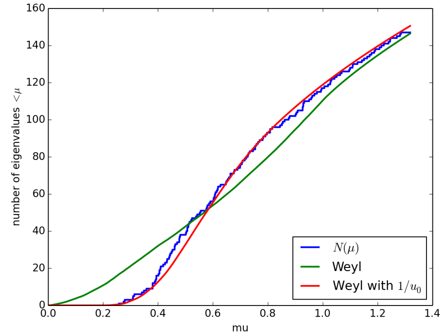


min $1/u$ vs E

Reading the localization landscape

- Modifying Weyl's law by replacing the true potential with the effective one predicts the density of states with astonishing accuracy.

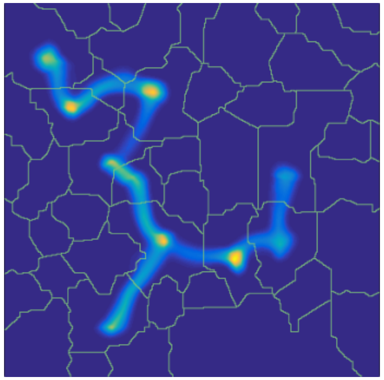
$$N(E) \approx Vol\left\{\xi^2 + \frac{1}{u} \leq E\right\}$$



Blue: reality; green: old Weyl; red: new Weyl

- It gives access to transport properties (hopping).

$$\langle\psi_1|e^{-i\vec{q}\cdot\vec{r}}|\psi_2\rangle \approx \int e^{-i\vec{q}\cdot\vec{r}} e^{\rho_{1,\frac{1}{u}}(\vec{r})+\rho_{2,\frac{1}{u}}(\vec{r})}\, d\vec{r}$$



Transport

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Mathematical Analysis

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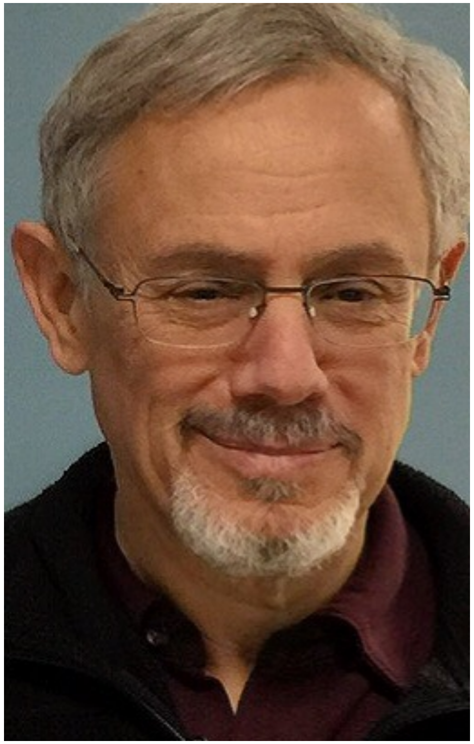

Mark Green,
UCLA (moderator)

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June 12*: *Number Theory: The Riemann Hypothesis*

July 10*: *Topology*

August 14*: *Algorithms for Threat Detection*

September 11: *Mathematical Analysis*

October 9: *Combinatorics*

November 13:
Why Machine Learning Works

December 11:
Mathematics of Epidemics

***** ***Recording posted***

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