



# MATHEMATICAL FRONTIERS

*The National  
Academies of* | SCIENCES  
ENGINEERING  
MEDICINE

[nas.edu/MathFrontiers](https://nas.edu/MathFrontiers)

Board on  
Mathematical Sciences & Analytics

# MATHEMATICAL FRONTIERS

## 2018 Monthly Webinar Series, 2-3pm ET

**February 13\*:**

*Mathematics of the Electric Grid*

**March 13\*:**

*Probability for People and Places*

**April 10\*:**

*Social and Biological Networks*

**May 8\*:**

*Mathematics of Redistricting*

**June 12\*:** *Number Theory: The Riemann Hypothesis*

**July 10\*:** *Topology*

**August 14\*:** *Algorithms for Threat Detection*

**September 11\*:** *Mathematical Analysis*

**October 9:** *Combinatorics*

**November 13:**

*Why Machine Learning Works*

**December 11:**

*Mathematics of Epidemics*

**\* Recording posted**

*Made possible by support for BMSA from the  
National Science Foundation Division of Mathematical Sciences and the  
Department of Energy Advanced Scientific Computing Research*

# MATHEMATICAL FRONTIERS

## Combinatorics



**Sara Billey,**  
University of Washington



**Jacques Verstraete,**  
University of California, San Diego



**Mark Green,**  
UCLA (moderator)

*View webinar videos and learn more about BMSA at [www.nas.edu/MathFrontiers](http://www.nas.edu/MathFrontiers)*



# MATHEMATICAL FRONTIERS

## Combinatorics



Sara Billey,  
University of Washington

*John Rainwater Faculty Fellow and  
Professor of Mathematics in the  
Department of Mathematics at the  
University of Washington*

## What is Combinatorics?

# What is Combinatorics?

Combinatorics is  
the nanotechnology of mathematics

This technology applies to problems on

- Existence
- Enumeration
- Optimization

of discrete structures taking into account constraints, patterns, preferences, and rules.

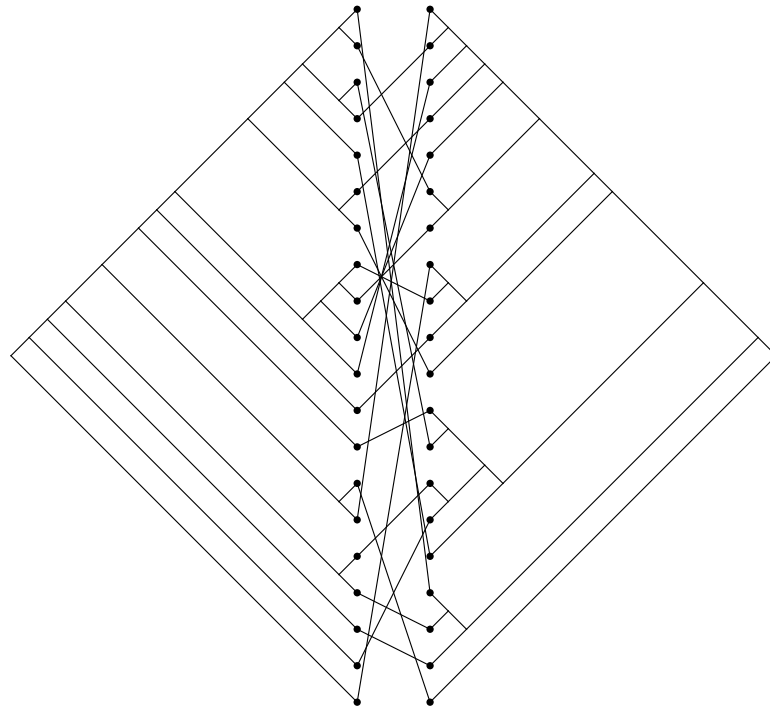
# Applications

In the past 100 years, combinatorics has revolutionized the way we think about problems in

- Biology
- Chemistry
- Computer Science
- Physics
- Industry
- Government
- Mathematics

# Examples

- The Stable Matching Algorithm
- Tanglegrams



# Example 1: Stable Matching

- In 1952, the **National Resident Matching Program (NRMP)** introduced an algorithm to match medical students to residency positions at hospitals in a way that respects the preferences of the students and hospitals without any there being any student-hospital pair who prefer each other over their assignment.
- In 1962, David Gale and Lloyd Shapley proved that the algorithm always produces an assignment which is simultaneously optimal for all students among all stable matchings.
- In 2012, Lloyd Shapley and Alvin Roth won the Nobel prize in Economics for their work realizing other non-monetary markets where the Stable Match Algorithm should be applied: kidney donation.
- **How does it work?**



# Stable Matching: How does it work?

- Students and hospitals each input a ranked list showing their preferences for the match.

Student Prfs	Boston	Houston	Seattle
Andrea	2	1	3
Lakshmi	1	3	2
Ming	2	1	3

Hospital Prfs	Andrea	Lakshmi	Ming
Boston	2	3	1
Houston	2	3	1
Seattle	3	2	1

# Stable Matching: How does it work?

- Students and hospitals each input a ranked list showing their preferences for the match.

Student Prfs	Boston	Houston	Seattle
Andrea	2	1	3
Lakshmi	1	3	2
Ming	2	1	3

Hospital Prfs	Andrea	Lakshmi	Ming
Boston	2	3	1
Houston	2	3	1
Seattle	3	2	1

# Stable Matching: How does it work?

- Students and hospitals each input a ranked list showing their preferences for the match.

Student Prfs	Boston	Houston	Seattle
Andrea	2	1	3
Lakshmi	1	3	2
Ming	2	1	3

Hospital Prfs	Andrea	Lakshmi	Ming
Boston	2	3	1
Houston	2	3	1
Seattle	3	2	1

# Stable Matching: How does it work?

- Students and hospitals each input a ranked list showing their preferences for the match.

Student Prfs	Boston	Houston	Seattle
Andrea	2	1	3
Lakshmi	1	3	2
Ming	2	1	3

One match:  
Andrea – Houston,  
Lakshmi – Boston,  
Ming – Seattle

Hospital Prfs	Andrea	Lakshmi	Ming
Boston	2	3	1
Houston	2	3	1
Seattle	3	2	1

Unstable pair:  
Ming – Houston  
prefer each other  
over their  
assignment

# Key Questions

- **Definition:** An assignment of students to hospitals is a **stable matching** if no student and hospital prefer each other over the one given by the assignment.
- **Existence Question:** Given any input preferences of  $n$  students and  $n$  hospitals, does a stable matching always exist?
- **Enumeration Question:** If so, how many stable matchings are there at most for  $n$  students and  $n$  hospitals?
- **Optimization Question:** Given any input preferences of  $n$  students and  $n$  hospitals, what is the best possible assignment for students? For hospitals?

# Stable Match Algorithm

- Each student “proposes” to their first choice of hospital.
- Then, hospitals reject all but their highest ranked proposal.
- Rinse, lather, repeat!

Student Prfs	Boston	Houston	Seattle
Andrea	2	1	3
Lakshmi	1	3	2
Ming	2	1	3

Hospital Prfs	Andrea	Lakshmi	Ming
Boston	2	3	1
Houston	2	3	1
Seattle	3	2	1



# Stable Match Algorithm

- Each student “proposes” to their first choice of hospital.
- Then, hospitals reject all but their highest ranked proposal.
- Rinse, lather, repeat!

Student Prfs	Boston	Houston	Seattle
Andrea	2	<del>1</del>	3
Lakshmi	1	3	2
Ming	2	1	3

Hospital Prfs	Andrea	Lakshmi	Ming
Boston	2	3	1
Houston	2	3	1
Seattle	3	2	1

# Stable Match Algorithm

- Each student “proposes” to their first choice of hospital.
- Then, hospitals reject all but their highest ranked proposal.
- Rinse, lather, repeat!

Student Prfs	Boston	Houston	Seattle
Andrea	2	<del>1</del>	3
Lakshmi	1	3	2
Ming	2	1	3

Hospital Prfs	Andrea	Lakshmi	Ming
Boston	2	3	1
Houston	2	3	1
Seattle	3	2	1

# Stable Match Algorithm

- Each student “proposes” to their first choice of hospital.
- Then, hospitals reject all but their highest ranked proposal.
- Rinse, lather, repeat!

Student Prfs	Boston	Houston	Seattle
Andrea	2	<del>1</del>	3
Lakshmi	<del>1</del>	3	2
Ming	2	1	3

Hospital Prfs	Andrea	Lakshmi	Ming
Boston	2	3	1
Houston	2	3	1
Seattle	3	2	1

# Stable Match Algorithm

- Each student “proposes” to their first choice of hospital.
- Then, hospitals reject all but their highest ranked proposal.
- Rinse, lather, repeat!

Student Prfs	Boston	Houston	Seattle
Andrea	2	<del>1</del>	3
Lakshmi	<del>1</del>	3	2
Ming	2	1	3

**Another match:**  
Andrea – Boston,  
Lakshmi – Seattle,  
Ming – Houston

Hospital Prfs	Andrea	Lakshmi	Ming
Boston	2	3	1
Houston	2	3	1
Seattle	3	2	1

**Stable!**  
No pair wants to  
disregard this  
assignment.

# Stable Match Algorithm

- **Theorem (Gale-Shapley):** For any input preferences by  $n$  students and  $n$  hospitals, the Stable Match Algorithm produces an assignment with no unstable pairs.
- **Theorem (Gale-Shapley):** Among all stable matchings of  $n$  students with  $n$  hospitals, this algorithm always finds the unique one that is best possible for every student.
- **Theorem (Gale-Shapley):** Among all stable matchings of  $n$  students with  $n$  hospitals, this algorithm always finds the unique one that is worst possible for every hospital.
- **Theorem (Knuth):** The Stable Match Algorithm runs in  $O(n^2)$  time on input from  $n$  students and  $n$  hospitals.

# Success of the Match Program

## NRMP Press Release from March 16, 2018

### Largest Match on Record

The 2018 Main Residency Match is the largest in NRMP history. A record-high 37,103 applicants submitted program choices for 33,167 positions, the most ever offered in the Match.

**Open Question:** What other problems can be solved by the Stable Match Algorithm?

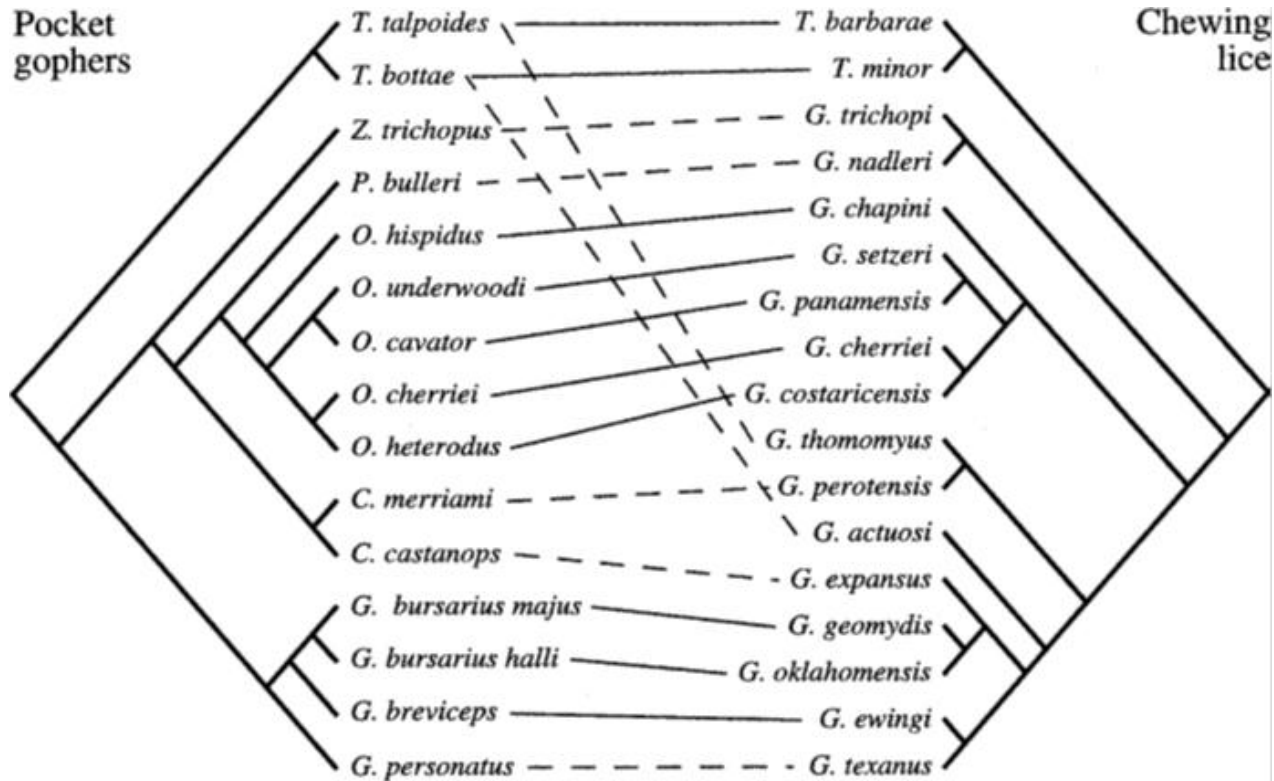
**Enumeration Question:** How many stable matchings exist for  $n$  students and  $n$  hospitals? (<https://oeis.org/A005154>)

See Also: The Stable Roommate Problem on Wikipedia

View webinar videos and learn more about BMSA at [www.nas.edu/MathFrontiers](http://www.nas.edu/MathFrontiers)



# Example 2: Tanglegrams



[https://evolutionnews.org/2012/01/parallel\\_host\\_a/](https://evolutionnews.org/2012/01/parallel_host_a/)

Definition: A **tanglegram** is a pair of binary trees with a matching between their leaves. They represent two phylogenetic trees of symbiotic organisms.

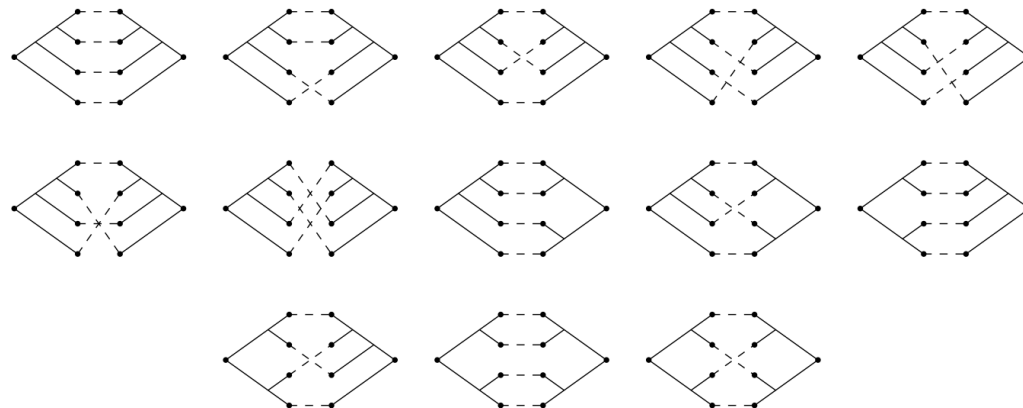
# Counting Tanglegrams

Erick Matsen, Arnold Kas and their team at the Fred Hutchinson Cancer Research Center study mathematical biology.

**Enumerative Question** (Matsen 2015):

Is there a nice formula to count the number of distinct tanglegrams with  $n$  leaves up to symmetries of the left tree and the right tree?

**Example:** for  $n=4$  there are 13 tanglegrams



# Counting Tanglegrams

**Enumerative Question** (Matsen 2015):

Is there a nice formula to count the number of distinct tanglegrams with  $n$  leaves up to symmetries of the left tree and the right tree?

Yes!

**Theorem** (Billey-Konvalinka-Matsen 2017): The number of tanglegrams of size  $n$  is

$$t_n = \sum_{\lambda} \frac{\prod_{i=2}^{\ell(\lambda)} \left(2(\lambda_i + \cdots + \lambda_{\ell(\lambda)}) - 1\right)^2}{z_{\lambda}},$$

summed over all binary partitions of  $n$ . The  $z$ -numbers are well known constants.

# Counting Tanglegrams

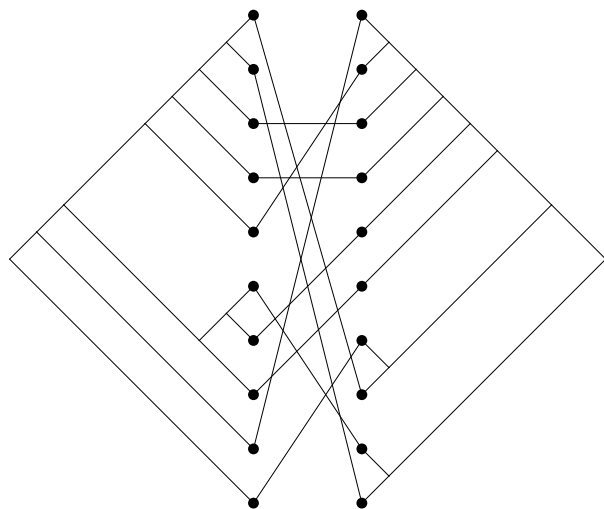
## Corollaries of the (Billey-Konvalinka-Matsen) Formula:

- The number of tanglegrams grows quickly:

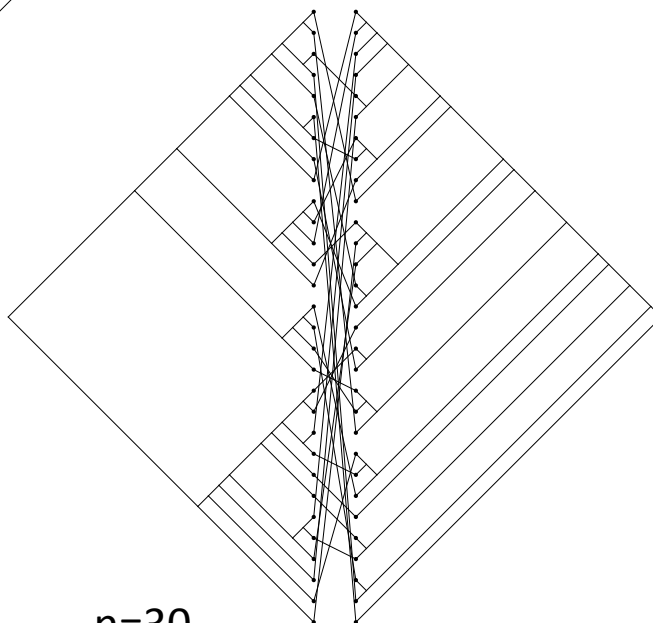
$$\frac{t_n}{n!} \sim \frac{e^{\frac{1}{8}} 4^{n-1}}{\pi n^3}$$

- We can compute the exact number of tanglegrams for  $n$  as large as 4000 using a recurrence relation derived from the formula.
- There is an algorithm to find a tanglegram of size  $n$  uniformly at random so we can study the average behavior of these objects.

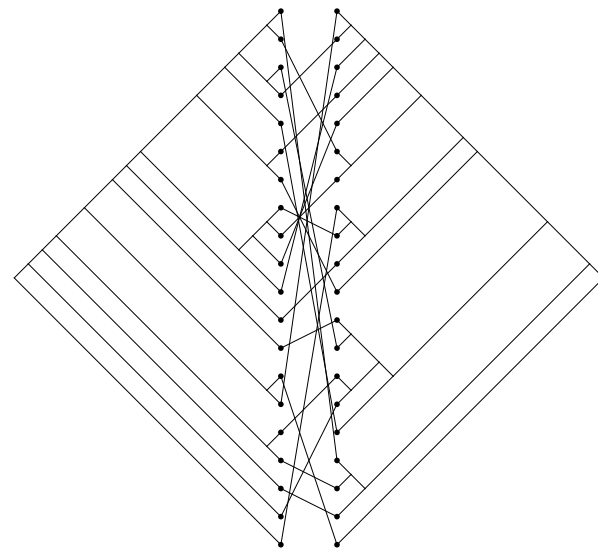
# Typical Tanglegrams



$n=10$



$n=30$



$n=20$



# Reprise

- Questions about existence, enumeration, and optimization of discrete structures appear in many science and industrial applications.
- Combinatorial algorithms for answering these questions have led to faster, cheaper, and more accurate solutions to many problems in our lives.
- Still many questions unanswered.
- Come, join, contribute to the Combinatorics Revolution!





# Resources and Acknowledgements

Many thanks to my colleagues on [bboard@math.uw](mailto:bboard@math.uw) for help on preparing this talk!

Thanks to all of you for listening and participating!

## Resources:

- “College Admissions and the Stability of Marriage”. David Gale and Lloyd Shapley. MAA Math Monthly 69, 9-14, 1962.
- “On the enumeration of tanglegrams and tangled chains” Sara Billey, Matjaz Konvalinka, Frederick A Matsen IV, J. Combin. Theory Ser. A, 146, pp 239--263, 2017.
- “Fingerprint Databases for Theorems” Sara Billey and Bridget Tenner. Notices of the AMS 60:8 (2013).
- “How to apply de Bruijn graphs to genome assembly” Phillip E C Compeau, Pavel A Pevzner, and Glenn Tesler. Nature Biotechnology 29, 987–991 (2011) <http://www.nature.com/nbt/journal/v29/n11/full/nbt.2023.html>

# MATHEMATICAL FRONTIERS

## Combinatorics



Jacques Verstraete,  
University of California, San Diego

*Graduate Vice-Chair and  
Professor of Mathematics in the  
Department of Mathematics at the  
University of California, San Diego*

## The Analysis of Finite Structures

# Introduction

- Combinatorics is the study of finite structures.
- The central objects in combinatorics are often motivated by concerns in other areas of science, especially theoretical computer science, bioinformatics and statistical physics.
- In turn, combinatorics has applications to other areas of mathematics, such as number theory, geometry, probability, and algebra.

# Introduction

- One of the most attractive topics in mathematics is the study of prime numbers.
- **Prime factorization:** every integer  $n > 1$  is a product of primes.

$$11111 = 41 \cdot 271 \quad 111111 = 3 \cdot 7 \cdot 11 \cdot 37 \quad 1111111 = 239 \cdot 4649$$

# Introduction

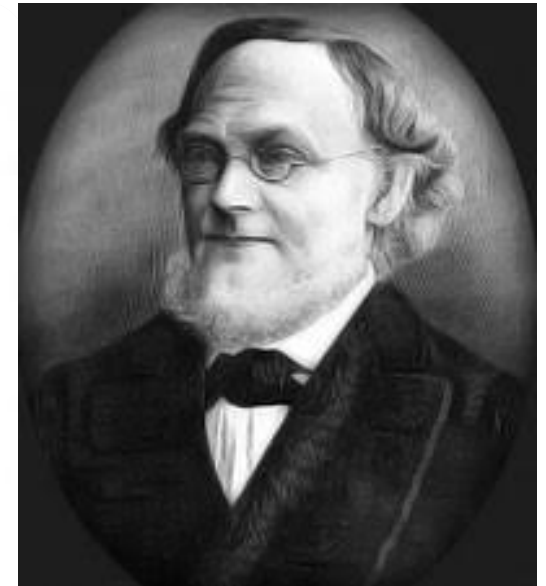
- Cryptography: **RSA** based on hardness of finding prime factorization. Variants underlie much of modern electronic security.
- **Primality testing**: polynomial time. (Agrawal, Kayal, Saxena, 2002)

# Introduction

- Goldbach's Conjecture<sup>1742</sup> : every even number larger than 2 is the sum of two primes.

haben, nicht bestanden, ob man aber schon nachherausfindet,  
man singt series lauter numeros unico modo in duo quadrata  
divisibiles geben auf solche Weise will ich auf eine conjecture  
hinzusetzen: daß jede Zahl welche aus zweyen numeros primis  
zusammengesetzt ist ein aggregatum so vieler numerorum  
primorum sey als man will /: die unitatem mit der zweyten  
hieß auf die congeriem omnium unitatum\*. zine fongue

$$4 = \begin{cases} 1+1+1+1 \\ 1+1+2 \\ 1+3 \end{cases} \quad 5 = \begin{cases} 2+3 \\ 1+1+3 \\ 1+1+1+2 \\ 1+1+1+1+1 \end{cases} \quad 6 = \begin{cases} 1+5 \\ 1+2+3 \\ 1+1+1+3 \\ 1+1+1+1+2 \\ 1+1+1+1+1+1 \end{cases} \quad \text{L.C.}$$



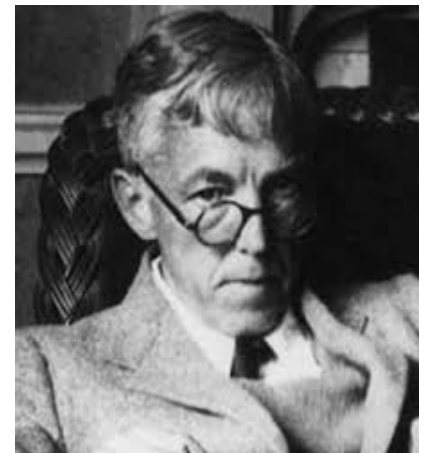


# Quasirandomness

- The **Extended Goldbach Conjecture** states that the number  $R(n)$  of representations of  $n$  as a sum of two primes satisfies:

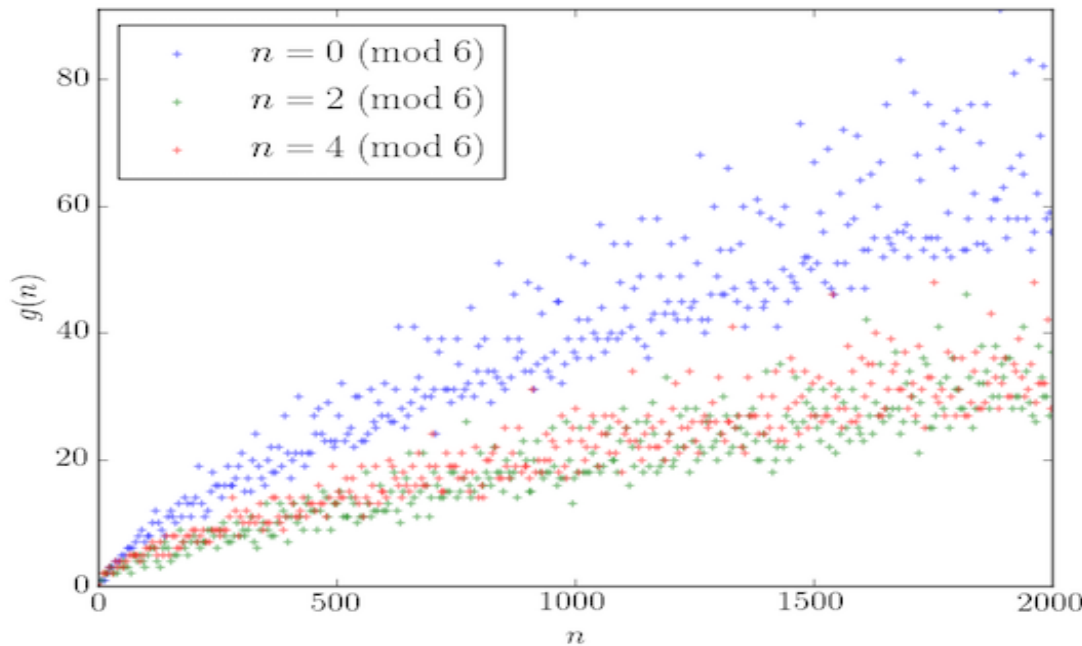
$$R(n) \sim 2\prod_2 \cdot \prod_{\substack{p>2 \\ p|n}} \frac{p-1}{p-2} \cdot \int_2^n \frac{1}{(\log x)^2} dx$$

(Hardy-Littlewood, 1923)



# Introduction

- Goldbach's Conjecture<sup>1742</sup> : every even number larger than 2 is the sum of two primes.



$$R(22) = 3$$

$$22 = 3 + 19$$

$$= 5 + 17$$

$$= 11 + 11$$

# Introduction

- Lagrange<sup>1770</sup> : for every positive integer  $k$  there exists a progression of  $k$  primes.

461	463	397	401	409	419
		467		479	487
9	601	607	541	617	557
	673	677	613	691	
	743		683	761	
9	811		751	829	769
	881	887	821	823	839
	953				
9	1021		1031	1039	907
	1091	1097	1033	1109	977
	1163		1103	1181	1049
9	1231	1237	1171	1249	1117
	1301	1307		1319	1187
	1373		1381	1321	1259
9		1447	1451	1459	1327
	1511		1523	1531	1399
	1583			1601	
		1657	1663	1667	1607
1721	1723		1733	1741	1609
			1801	1811	1747
	1861	1867	1871	1877	
1931	1933		1873	1879	1889
	2003			1949	
		2011		1951	
		2081	2083	2087	2027
		2089	2089	2089	2029
		2093	2093	2093	2093
		2099	2099	2099	2099
		2101	2101	2101	2101
		2107	2107	2107	2107
		2111	2111	2111	2111
		2113	2113	2113	2113
		2117	2117	2117	2117
		2123	2123	2123	2123
		2129	2129	2129	2129
		2131	2131	2131	2131
		2137	2137	2137	2137
		2141	2141	2141	2141
		2143	2143	2143	2143
		2147	2147	2147	2147
		2153	2153	2153	2153
		2159	2159	2159	2159
		2161	2161	2161	2161
		2167	2167	2167	2167
		2171	2171	2171	2171
		2173	2173	2173	2173
		2177	2177	2177	2177
		2183	2183	2183	2183
		2189	2189	2189	2189
		2191	2191	2191	2191
		2197	2197	2197	2197
		2201	2201	2201	2201
		2203	2203	2203	2203
		2207	2207	2207	2207
		2213	2213	2213	2213
		2219	2219	2219	2219
		2221	2221	2221	2221
		2227	2227	2227	2227
		2231	2231	2231	2231
		2237	2237	2237	2237
		2243	2243	2243	2243
		2251	2251	2251	2251
		2257	2257	2257	2257
		2261	2261	2261	2261
		2267	2267	2267	2267
		2271	2271	2271	2271
		2273	2273	2273	2273
		2277	2277	2277	2277
		2281	2281	2281	2281
		2283	2283	2283	2283
		2287	2287	2287	2287
		2291	2291	2291	2291
		2293	2293	2293	2293
		2297	2297	2297	2297
		2301	2301	2301	2301
		2303	2303	2303	2303
		2307	2307	2307	2307
		2311	2311	2311	2311
		2313	2313	2313	2313
		2317	2317	2317	2317
		2321	2321	2321	2321
		2323	2323	2323	2323
		2327	2327	2327	2327
		2331	2331	2331	2331
		2333	2333	2333	2333
		2337	2337	2337	2337
		2341	2341	2341	2341
		2343	2343	2343	2343
		2347	2347	2347	2347
		2351	2351	2351	2351
		2353	2353	2353	2353
		2357	2357	2357	2357
		2361	2361	2361	2361
		2363	2363	2363	2363
		2367	2367	2367	2367
		2371	2371	2371	2371
		2373	2373	2373	2373
		2377	2377	2377	2377
		2381	2381	2381	2381
		2383	2383	2383	2383
		2387	2387	2387	2387
		2391	2391	2391	2391
		2393	2393	2393	2393
		2397	2397	2397	2397
		2401	2401	2401	2401
		2403	2403	2403	2403
		2407	2407	2407	2407
		2411	2411	2411	2411
		2413	2413	2413	2413
		2417	2417	2417	2417
		2421	2421	2421	2421
		2423	2423	2423	2423
		2427	2427	2427	2427
		2431	2431	2431	2431
		2433	2433	2433	2433
		2437	2437	2437	2437
		2441	2441	2441	2441
		2443	2443	2443	2443
		2447	2447	2447	2447
		2451	2451	2451	2451
		2453	2453	2453	2453
		2457	2457	2457	2457
		2461	2461	2461	2461
		2463	2463	2463	2463
		2467	2467	2467	2467
		2471	2471	2471	2471
		2473	2473	2473	2473
		2477	2477	2477	2477
		2481	2481	2481	2481
		2483	2483	2483	2483
		2487	2487	2487	2487
		2491	2491	2491	2491
		2493	2493	2493	2493
		2497	2497	2497	2497
		2501	2501	2501	2501
		2503	2503	2503	2503
		2507	2507	2507	2507
		2511	2511	2511	2511
		2513	2513	2513	2513
		2517	2517	2517	2517
		2521	2521	2521	2521
		2523	2523	2523	2523
		2527	2527	2527	2527
		2531	2531	2531	2531
		2533	2533	2533	2533
		2537	2537	2537	2537
		2541	2541	2541	2541
		2543	2543	2543	2543
		2547	2547	2547	2547
		2551	2551	2551	2551
		2553	2553	2553	2553
		2557	2557	2557	2557
		2561	2561	2561	2561
		2563	2563	2563	2563
		2567	2567	2567	2567
		2571	2571	2571	2571
		2573	2573	2573	2573
		2577	2577	2577	2577
		2581	2581	2581	2581
		2583	2583	2583	2583
		2587	2587	2587	2587
		2591	2591	2591	2591
		2593	2593	2593	2593
		2597	2597	2597	2597
		2601	2601	2601	2601
		2603	2603	2603	2603
		2607	2607	2607	2607
		2611	2611	2611	2611
		2613	2613	2613	2613
		2617	2617	2617	2617
		2621	2621	2621	2621
		2623	2623	2623	2623
		2627	2627	2627	2627
		2631	2631	2631	2631
		2633	2633	2633	2633
		2637	2637	2637	2637
		2641	2641	2641	2641
		2643	2643	2643	2643
		2647	2647	2647	2647
		2651	2651	2651	2651
		2653	2653	2653	2653
		2657	2657	2657	2657
		2661	2661	2661	2661
		2663	2663	2663	2663
		2667	2667	2667	2667
		2671	2671	2671	2671
		2673	2673	2673	2673
		2677	2677	2677	2677
		2681	2681	2681	2681
		2683	2683	2683	2683
		2687	2687	2687	2687
		2691	2691	2691	2691
		2693	2693	2693	2693
		2697	2697	2697	2697
		2701	2701	2701	2701
		2703	2703	2703	2703
		2707	2707	2707	2707
		2711	2711	2711	2711
		2713	2713	2713	2713
		2717	2717	2717	2717
		2721	2721	2721	2721
		2723	2723	2723	2723
		2727	2727	2727	2727
		2731	2731	2731	2731
		2733	2733	2733	2733
		2737	2737	2737	2737
		2741	2741	2741	2741
		2743	2743	2743	2743
		2747	2747	2747	2747
		2751	2751	2751	2751
		2753	2753	2753	2753
		2757	2757	2757	2757
		2761	2761	2761	2761
		2763	2763	2763	2763
		2767	2767	2767	2767
		2771	2771	2771	2771
		2773	2773	2773	2773
		2777	2777	2777	2777
		2781	2781	2781	2781
		2783	2783	2783	2783
		2787	2787	2787	2787
		2791	2791	2791	2791
		2793	2793	2793	2793
		2797	2797	2797	2797
		2801	2801	2801	2801
		2803	2803	2803	2803
		2807	2807	2807	2807
		2811	2811	2811	2811
		2813	2813	2813	2813
		2817	2817	2817	2817
		2821	2821	2821	2821
		2823	2823	2823	2823
		2827	2827	2827	2827
		2831	2831	2831	2831
		2833	2833	2833	2833
		2837	2837	2837	2837
		2841	2841	2841	2841
		2843	2843	2843	2843
		2847	2847	2847	2847
		2851	2851	2851	2851
		2853	2853	2853	2853
		2857	2857	2857	2857
		2861	2861	2861	2861
		2863	2863	2863	2863
		2867	2867	2867	2867
		2871	28		

# The probabilistic method

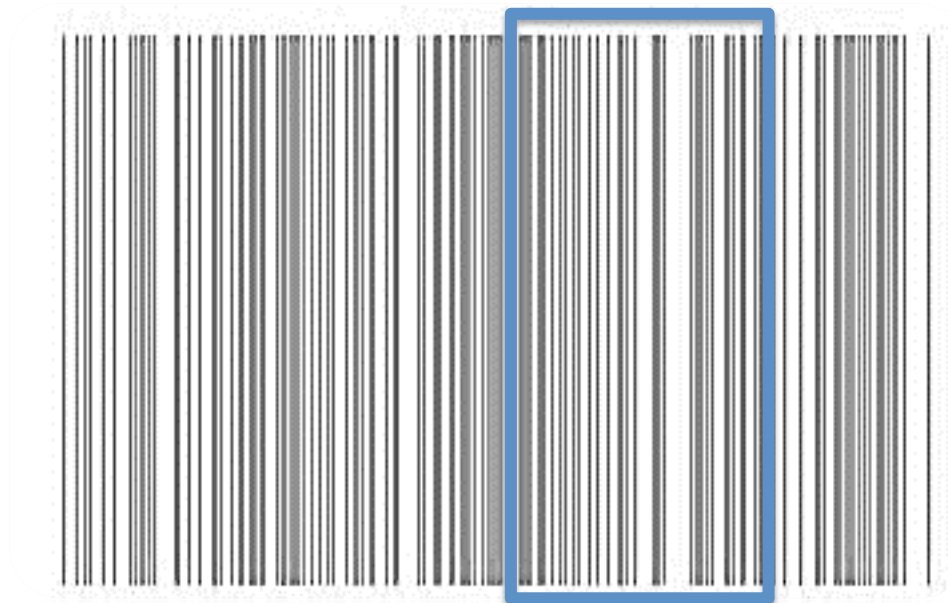
- In many areas of mathematics, one is required to construct a structure under a prescribed list of constraints, or at least prove its existence.
- The probabilistic method was introduced by Paul Erdős over fifty years ago.
- The next examples illustrate one of the organizing principles of the method:



if it seems likely that the structure we want is roughly uniform,  
then a random example is worth trying.

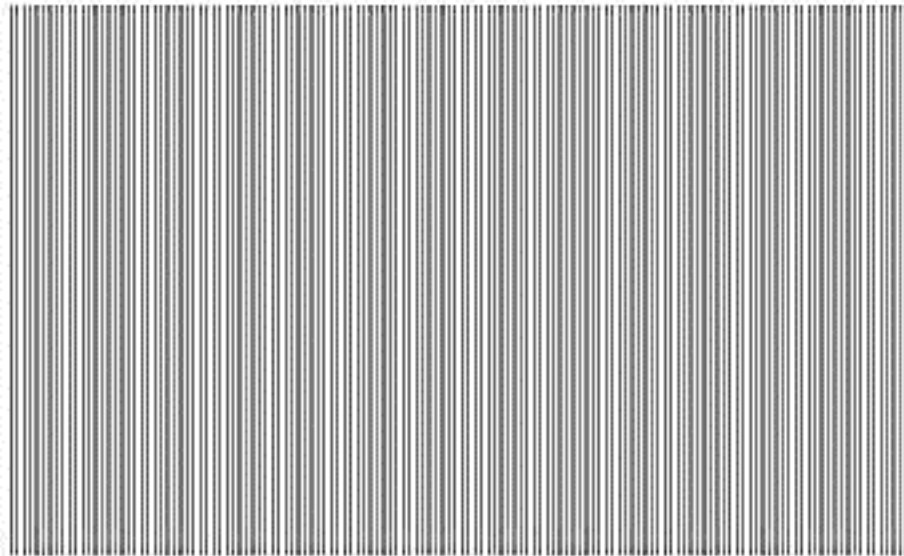
# Randomness versus Structure

- Suppose we select a random set of numbers from 1 to  $n$ , where each number is selected independently with probability  $p$ .
- We would expect every interval of  $m$  consecutive numbers contains about  $pm$  selections.



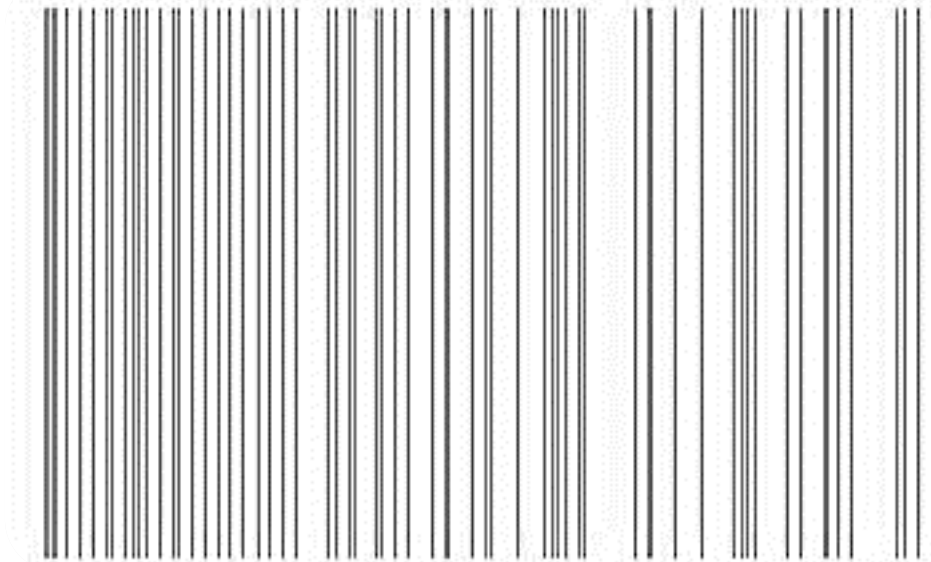
# Randomness versus Structure

- The set of even numbers, on the other hand, should be considered to be “structured”.
- More generally, any union of few arithmetic progressions should be considered “structured”



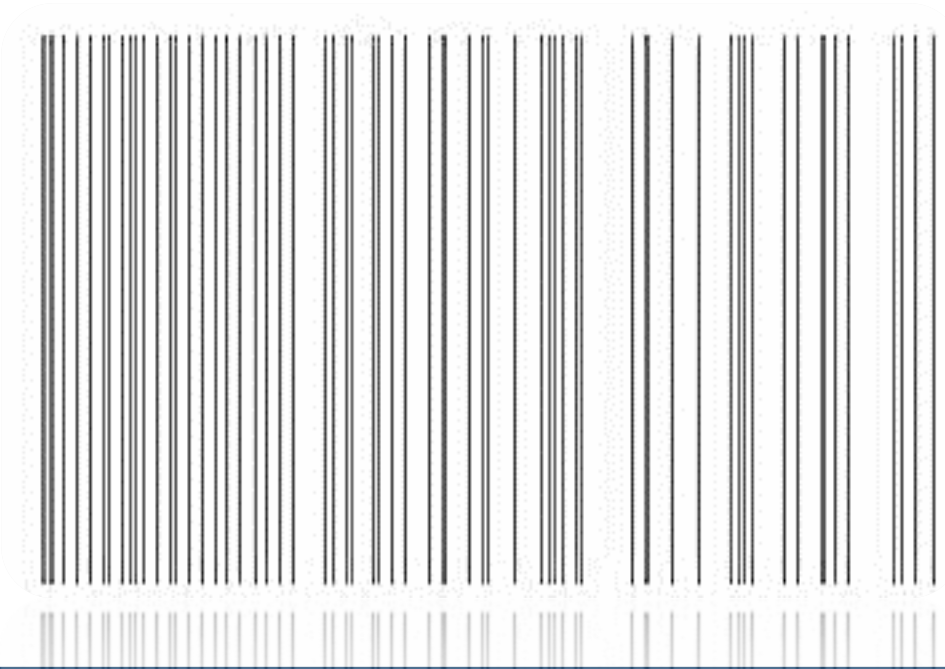
# Randomness versus Structure

- For instance, consider the set of prime numbers.
- According to the **Prime Number Theorem**, there are roughly  $\frac{n}{\log n}$  primes less than  $n$ .



# Randomness versus Structure

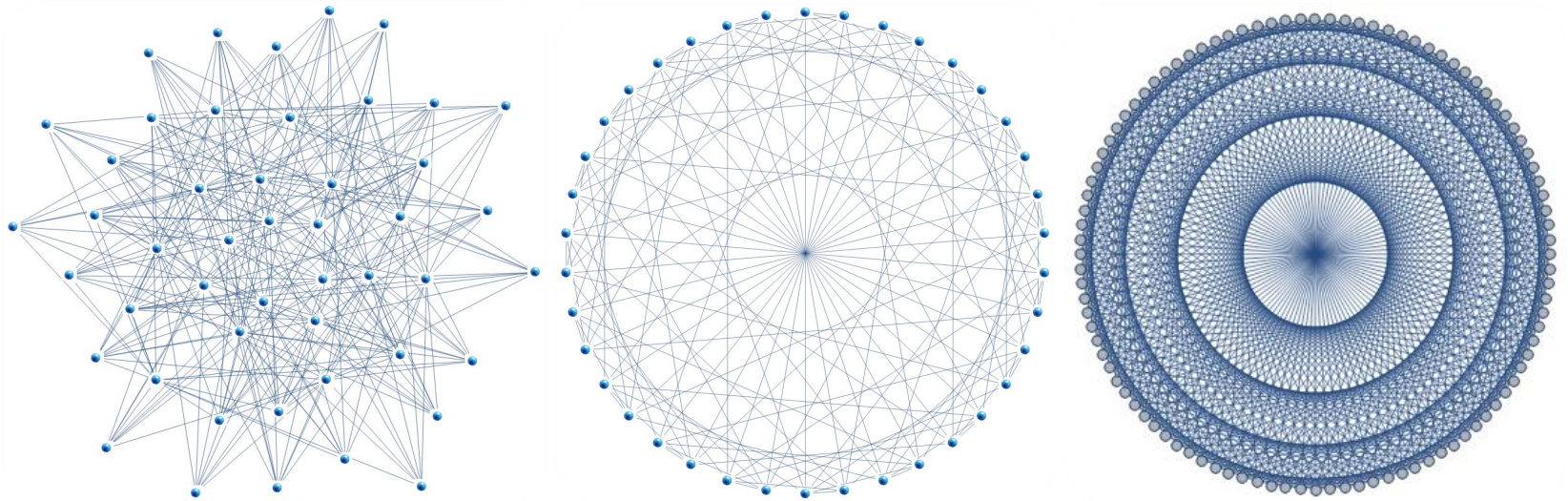
- **Cramér's Conjecture** : There is a prime between  $n$  and about  $n + (\log n)^2$  for every  $n$ . (Cramér, 1936)





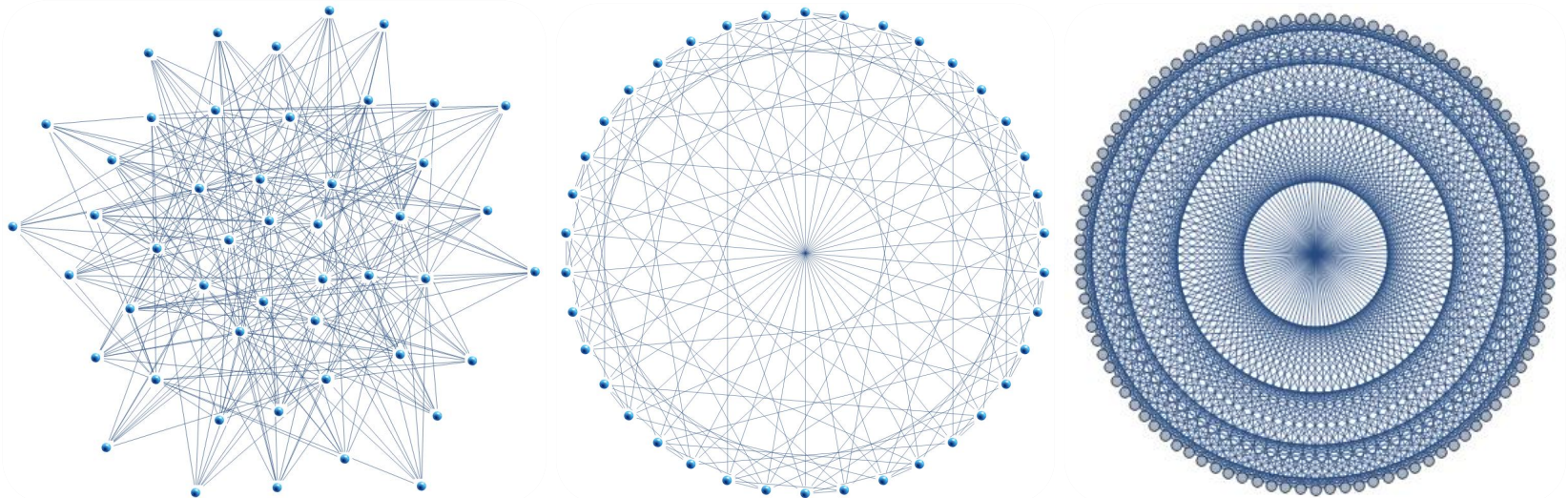
# Randomness versus Structure

- A **graph** is a set of **vertices** / **nodes** together with a set of pairs of vertices called **edges**.
- These are fundamental objects in combinatorics.



# Randomness versus Structure

- When is a graph “random”?
- Place edges randomly and independently with probability  $p$ .



# Quasirandomness

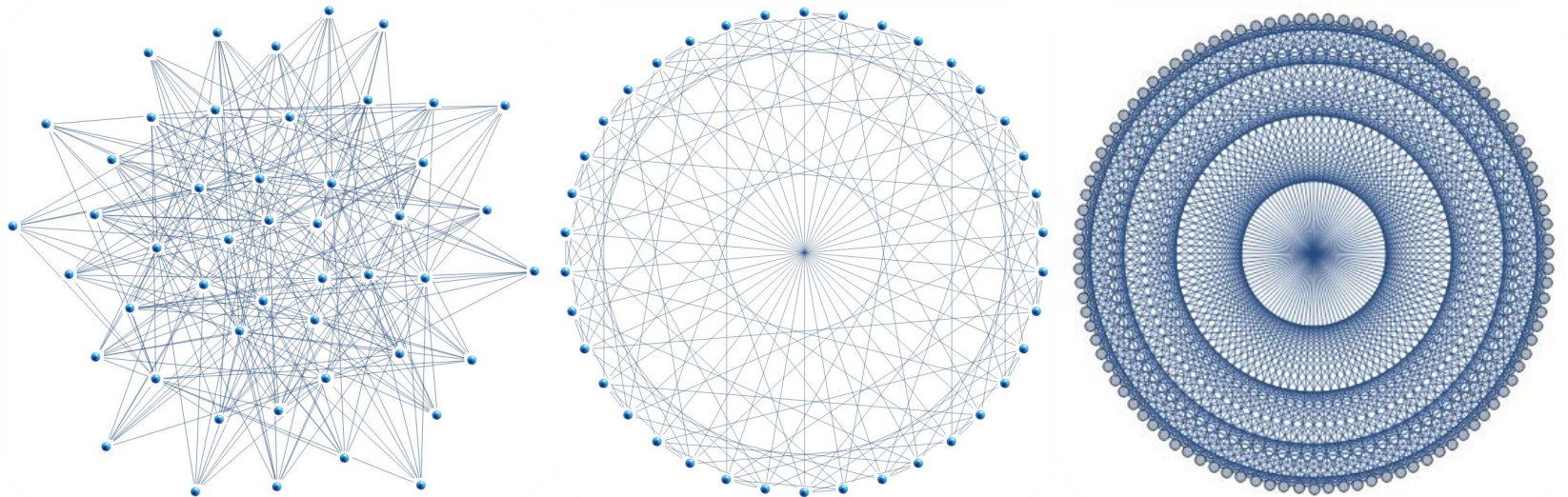
- Given any set  $X$  of vertices, we expect  $p \binom{|X|}{2}$  edges of the graph to lie inside  $X$ .
- We call an  $n$ -vertex graph of density  $p$  an  $\varepsilon$ -quasirandom graph if for every set  $X$

$$\left| e(X) - p \binom{|X|}{2} \right| < \varepsilon p n^2$$



# Randomness versus Structure

- When is a graph “quasirandom”?



# Quasirandomness

- How to tell if a graph is random? Using **spectral theory** of the graph matrices.
- **Expander Mixing Lemma (Alon, 1986)**

$$\left| e(X) - p \binom{|X|}{2} \right| \leq \Pi |X|$$

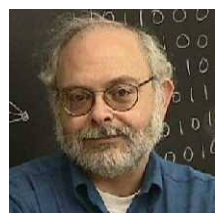


# Quasirandomness

- How to tell if a graph is random? Counting quadrilaterals.
- Thomason (1987), Chung-Graham-Wilson (1991)

A graph with  $n$  vertices and density  $p$  is  $\varepsilon$ -quasirandom if and only if the number of quadrilaterals in the graph is at most  $(1 + \varepsilon^4)(pn)^4$

- Quasirandom graphs appear frequently in applications, for example in coding and information theory (expander graphs).



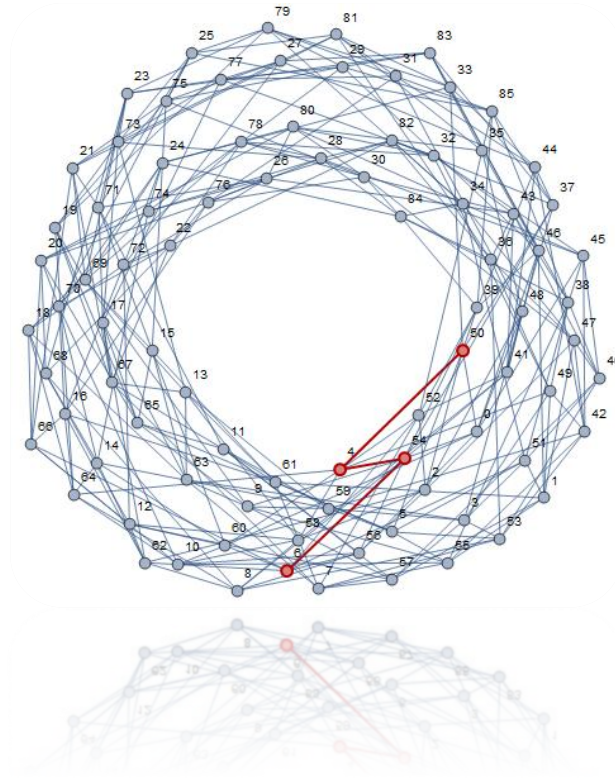
# Quasirandomness

- We can use graphs to find arithmetic progressions in sets of integers.
- Szemerédi's Theorem (1975)  
Every set of integers positive density contains arbitrarily long progressions.



# Quasirandomness

- The arithmetic progression  $\{3,5,7\}$

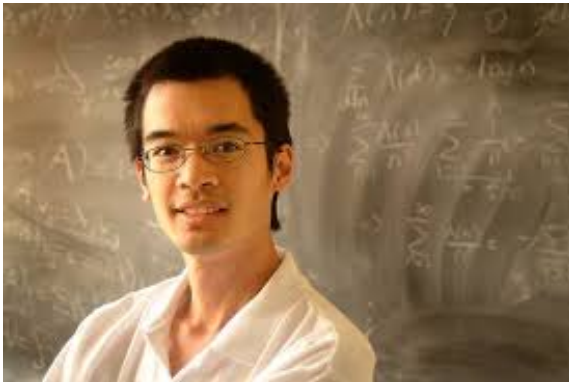




# Breakthroughs

- Theorem. (Green-Tao Theorem, 2006)

The primes contain arbitrarily long arithmetic progressions.



# Conclusion

- Combinatorics has burgeoned into a fundamental part of modern mathematics, establishing many connections and applications to many other areas of science.
- We discussed a general modern theme in combinatorics, which is to distinguish between **randomness** and **structure** in combinatorial objects.
- The **probabilistic method** has led to a number of recent breakthroughs.

# MATHEMATICAL FRONTIERS

## Combinatorics



**Sara Billey,**  
University of Washington



**Jacques Verstraete,**  
University of California, San Diego



**Mark Green,**  
UCLA (moderator)

# MATHEMATICAL FRONTIERS

## 2018 Monthly Webinar Series, 2-3pm ET

**February 13\*:**

*Mathematics of the Electric Grid*

**March 13\*:**

*Probability for People and Places*

**April 10\*:**

*Social and Biological Networks*

**May 8\*:**

*Mathematics of Redistricting*

**June 12\*:** *Number Theory: The Riemann Hypothesis*

**July 10\*:** *Topology*

**August 14\*:** *Algorithms for Threat Detection*

**September 11\*:** *Mathematical Analysis*

**October 9:** *Combinatorics*

**November 13:**

*Why Machine Learning Works*

**December 11:**

*Mathematics of Epidemics*

**\* Recording posted**

*Made possible by support for BMSA from the  
National Science Foundation Division of Mathematical Sciences and the  
Department of Energy Advanced Scientific Computing Research*